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Proclus Hypothesis

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

This article substantiates a new look on Euclid's *Elements* and mathematics history, based on *Proclus hypothesis*. Proclus hypothesis answers the question about Euclid's goal for writing his *Elements.* Two Greek mathematical achievements underlie Proclus hypothesis: the *golden ratio*, described in the Books II, VI and XIII, and *Platonic solids*, described in the final Book XIII of the *Elements*. As the *golden ratio* and *Platonic solids* expressed the Universe harmony in Greek science, it follows from Proclus hypothesis that the main Euclid's goal in his *Elements* is to embody Pythagoras & Plato's "Ideas of the Universe Harmony." Euclid's *Elements* are historically the first version of the *Mathematics of Harmony* as one of the main directions in mathematics development. This approach overturns our understanding of Euclid's *Elements* and mathematics history starting from Euclid. The article presents a general interest for all mathematicians, math teachers, mathematics students, and for all science representatives, who are interested for new ideas in the history of mathematics.

Keywords: Euclid's elements; Proclus hypothesis; the golden ratio; platonic solids; fullerenes; quasi-crystals; the "golden" number theory; Hilbert's fourth problem; the fine-structure constant problem.

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1 Introduction

Pythagorean “Mathem’s.” By studying the initial sources of mathematics origin, we inevitably come to Pythagoras and his doctrine called *Pythagoreanism* [1,2]. Pythagoras was the first thinker who according to legend called himself a *philosopher* that is *"a lover of wisdom".* He first called the Universe by *Cosmos* that means *"beautiful order"*. The subject of his doctrine was the world as a harmonious whole, which is subjected to the laws of harmony and number.

As it is highlighted in [1,2], according to tradition, Pythagoreans were divided into two separate schools of thought, the *mathēmatikoi* (Greek for *"teachers"* or *“mathematicians”*) and the *akousmatikoi* (Greek for *"listeners"*). The *akousmatikoi* (listeners) dealt with religious and ritual part of Pythagorean doctrine, *mathēmatikoi* (mathematicians) dealt with studies of the four *Pythagorean “Mathem’s": arithmetic*, *geometry*, *harmonics* and *spherics*.

Pythagorean philosophy had a marked impact on Plato’s studies. Of particular interest is application of the *Platonic Solids*, derived by Plato from the Pythagorean theories of geometry and numbers. Starting from Plato, Platonic solids had passed a red thread through the history of mathematics and theoretical natural sciences, to the works of Euclid, Johannes Kepler, Felix Klein and from them to modern *quasi-crystals* [3] and *fullerenes* [4].

Unfortunately, in the course of historical development, one of the Pythagorean "Mathem’s" (*“harmonics”*) disappeared from mathematics, although an interest in "Harmony Problem" was preserved in other areas of human intellect, particularly in philosophy and theoretical natural sciences.

Euclid’s *Elements.* Euclid's *Elements* are the next step in the development of mathematics. It is known, that Euclid's *Elements* had an enormous influence on the development of not only geometry and mathematics, but of the entire science in general. As it is emphasized in Wikipedia [5], “*the Elements are still considered a masterpiece in the application of logic to mathematics. In historical context, it has proven enormously influential in many areas of science. Scientists* ***Nicolaus Copernicus****,* ***Johannes Kepler****,* ***Galileo Galilei****, and Sir* ***Isaac Newton*** *were all influenced by the Elements, and applied their knowledge of it to their work. Mathematicians and philosophers, such as* ***Bertrand Russell****,* ***Alfred North Whitehead****, and* ***Baruch Spinoza****, have attempted to create their own foundational "Elements" for their respective disciplines, by adopting the axiomatized deductive structures that Euclid's work introduced”*.

The Mathematics of Harmony. Author’s book “The Mathematics of Harmony. From Euclid to Contemporary Mathematics and Computer Science” (World Scientific, 2009) [6] can be considered as revival of Pythagoras&Plato’s “harmonic” ideas in contemporary mathematics and science. For the first time the conception of the Harmony Mathematics has been introduced by the author in the 1998 article [7] published in the book “Applications of Fibonacci Numbers” after author’s speech made at the 7th International Conference “Fibonacci Numbers and Their Applications” (Austria, Graz, July 1996). Since 1996, the Mathematics of Harmony and its applications in modern science became the main purpose of author’s researches.

The story of the book [6] begins with the author's 1977, 1979, 1984 books [8-10]. Although these books appeared as result of solving applied problems (optimal algorithms for measuring (analog-to-digital conversion) [8,9] and new arithmetical basis of computers [10]), in fact, they went off far beyond these applications and touched the basis of mathematics (new mathematical theory of measuring and new methods of number representation).

The book [6] consists of three parts:

Part I. Classical Golden Mean, Fibonacci Numbers, and Platonic Solids
Part II. Mathematics of Harmony
Part III. Applications in Computer Science

Two important applications of the Mathematics of Harmony follow from the book [6]: *Digital Metrology* (optimal algorithms of analog-to-digital conversion) and *Computer Science* (new positional methods of number representation and following from them new computer arithmetic's, *Fibonacci* and *"Golden" arithmetic's* [8-10]) are the first areas of applications. These applications continue evolving in the 21st century and they pass already from the stage of theoretical studies to the stage of engineering designing (Fibonacci computers). Publication of the article "Brousentsov's ternary principle, Bergman's number system and ternary mirror-symmetrical arithmetic" [11] is a confirmation of this direction. The article [11], published in the prestigious "Computer Journal" (British Computer Society), aroused great interest among readers. Donald Knuth became the first prominent computer specialist, who congratulated the author with this publication.

The second direction of the Mathematics of Harmony is connected with new classes of hyperbolic functions, the *hyperbolic Fibonacci and Lucas functions,* which have direct relation to non-Euclidean geometry [12-15].

Proclus hypothesis. By discussing Euclid's Elements [5], we always try to find the answers to the following questions:

1. What was the main Euclid's goal to write his *Elements*?
2. Why Euclid has introduced the *golden ratio* (Proposition II.11) and how he used this notion in his *Elements*?
3. Are there connection between *Pythagorean "Mathem's"* and Euclid's *Elements*?

The author found the answers to these questions in the book of the Belarusian philosopher Eduard Soroko "Structural Harmony of Systems" [16]. Soroko refers to the astonishing hypothesis of the Greek philosopher Proclus [17], one of the first Euclid's commentators. The book [16] presents *Proclus' hypothesis* as follows:

> *"Euclid wrote his Elements not for the purpose of presentation of geometry itself, but for the purpose to set forth a complete systematic geometric theory of the five Platonic solids, simultaneously describing some of the latest achievements of mathematics".*

This quote has become one of the guiding ideas in the substantiating of the deep connection between the *Mathematics of Harmony* [6] and Euclid's *Elements* [5].

The author presents his views on *Proclus' hypothesis* and the *Mathematics of harmony* in the articles [18-20].

The present article is a continuation and development of the articles [18-20]. Here, the following question appears. What is the purpose of the publication of the present article, if the articles on similar topic [18-20] have been published? Why the author decided to turn once again to the justification of *Proclus' hypothesis* and the *Mathematics of Harmony*?

First of all, we must emphasize that *Proclus hypothesis* is unusual and revolutionary hypothesis in mathematics history. It overturns our ideas about Euclid's *Elements* and the history of mathematics. On the other hand, the *Mathematics* of *Harmony* [6] is the new-found scientific discipline (2009) and its role in modern science isn't recognized fully until now.

However, the modern *Mathematics of Harmony* is developing today very fast, and its new results, published in the *British Journal of Mathematics and Computer Science* [21,22] and in the *Physical Sciences International Journal* [23], are lifting the *Mathematics of Harmony* on a new scientific level, the level of the *Millennium Problems*. That is why, the publication of this article, which represents original resume of the important stage in the development of the *Mathematics of Harmony*, set forth in the articles [18-20], is entirely appropriate, and can attract attention of readers to the *Mathematics of Harmony* [6].

2 The Golden Ratio in Euclid's Elements

Let us consider the following Proposition 11 from the Book II of Euclid's *Elements.*

Proposition II.11. *Divide a given line segment* $AD=a+b$ *into two unequal parts* $AF=b$ *and* $FD=a$ *so that the area of the square, which is built on the larger segment* $AF=b$ *would be equal to the area of the rectangle, which is built on the segment* $AD=a+b$ *and the smaller segment* $FD=a$.

Depict this problem geometrically (Fig. 1).

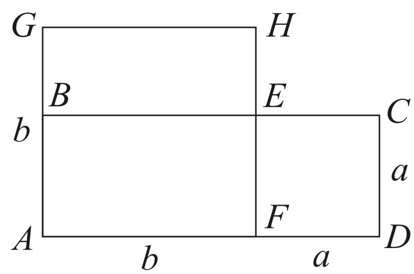


Fig. 1. A division of a line segment in extreme and mean ratio (the golden ratio)

A geometric task, given by Proposition II.11, is called a *division of line segment* $AD=a+b$ *in extreme and mean ratio* or the *golden ratio.*

Note that the geometric figure *AGHF* is the quadrate with the area b^2 and the geometric figure *ABCD* is the rectangle with the area $a\times(a+b)$. The Proposition II.11 is reduced to the task to build the quadrate *AGHF* and the rectangular *ABCD* with equal areas:

$$b^2=a\times(a+b). \tag{1}$$

If we divide both parts of (1) at first by *a*, and then by *b*, we get the following proportion:

$$x=\frac{b}{a}=\frac{a+b}{b} \tag{2}$$

what corresponds to the following geometric proportion:

$$x=\frac{AF}{FD}=\frac{AD}{AF} \tag{3}$$

Then, taking into consideration that $AD=AF+FD$, the proportion (3) can be written as follows:

$$x=\frac{AF+FD}{AF}=1+\frac{FD}{AF}=1+\frac{1}{\frac{AF}{FD}}=1+\frac{1}{x}\cdot \tag{4}$$

The following algebraic equation follows from (4):

$$x^2-x-1=0\ . \tag{5}$$

From the "physical meaning" of the proportion (3), it implies that we should use the positive root of the equation (5). We name this root the *golden ratio* and denote it by Φ:

$$\Phi = \frac{1+\sqrt{5}}{2} . \tag{6}$$

The following remarkable identities for the *golden ratio* (6) follow from the algebraic equation (5):

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+...}}}} . \tag{7}$$

$$\Phi = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+...}}}} . \tag{8}$$

The expression (8) is of deep mathematic sense. The Russian mathematicians A.Y. Khinchin [24] and N.N. Vorobyov [25] drew attention on the fact that the expression (8) singles out the *golden ratio* (6) among other irrational numbers, because according to (8) the *golden ratio* is approximated by rational numbers the most slowly. This fact emphasizes the *uniqueness* of the *golden ratio* among other irrational numbers from the point of view of continued fractions.

3 Platonic Solids in Euclid's *Elements*

The Greeks made the first attempt to "mathematize harmony," that is, to express the harmony in numerical and geometric form. According to Eduard Soroko [16], *"the ancient Greeks associated the idea about the "universal harmony" of the Universe with its implementation in the Platonic solids."* In other words, the regular polyhedra, called *Platonic solids* after the prominent Greek philosopher Plato (Fig. 2), were considered as the indicators of the geometric harmony of the Universe.

There are surprising geometrical connections between Platonic solids. For example, the *cube* and the *octahedron* (see Fig. 2) are *dual* one to another, that is, they can be obtained one from another, if we take the centroids of the faces of the first figure as the vertexes of another one and conversely. Similarly, the *icosahedron* is dual to the *dodecahedron* (see Fig. 2). The *tetrahedron* (Fig. 2) is *dual* to itself.

The *dodecahedron* and the *dual* to it *icosahedron* (see Fig. 2) take a special place among the Platonic solids. First of all, we note that the geometry of the *dodecahedron* and the *icosahedron* relates directly to the *golden ratio*. Indeed, all faces of the dodecahedron (Fig. 2) are *regular pentagons*, based on the *golden ratio*. If we look closely at the icosahedron in Fig. 2, we can see that in each of its vertices, the five triangles come together and their outer sides form the *regular pentagon* based on the *golden ratio*. Even these facts are enough to make sure that the *golden ratio* plays a crucial role in the geometric construction of these Platonic solids.

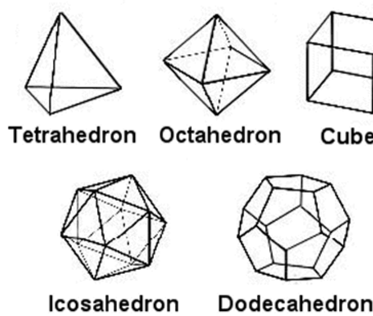


Fig. 2. Platonic solids

(Wikipedia, the free encyclopedia, from https://www.google.ca/?gws_rd=ssl#q=platonic+solids+images)

But there are the deeper mathematical confirmations of the fundamental role of the golden ratio in the icosahedron and the dodecahedron. It is known that the Platonic solids have three specific spheres. The first (*inner*) sphere is inscribed into the Platonic solid and touches its faces. We denote the radius of the inner sphere by R_i. The second or *middle* sphere of the Platonic solid touches its ribs. We denote the radius of the middle sphere by R_m. Finally, the third (*outer*) sphere is described around the Platonic solid and passes through its vertices. We denote its radius by R_c. In geometry, it is proved that the values of the radii of these spheres for the dodecahedron and the icosahedron with an edge of unit length are expressed through the golden ratio (see Table below).

	R_c	R_m	R_i
Icosahedron	$\frac{1}{2}\Phi\sqrt{3-\Phi}$	$\frac{1}{2}\Phi$	$\frac{\frac{1}{2}\Phi^2}{\sqrt{3}}$
Dodecahedron	$\frac{\Phi\sqrt{3}}{2}$	$\frac{\Phi^2}{2}$	$\frac{\Phi^2}{2\sqrt{3-\Phi}}$

Note that the ratio of the radii $\frac{R_c}{R_i}=\frac{\sqrt{3(3-\Phi)}}{\Phi}$ is the same for the icosahedron and the dodecahedron. Thus, if the dodecahedron and the icosahedron have the same inner spheres, their outer spheres are equal. This is a reflection of the "hidden harmony" of the dodecahedron and the icosahedron.

4 Proclus Hypothesis and a New Look on Euclid's *Elements*

As follows from Wikipedia [17], **Proclus** was born February 8, 412 AD in Constantinople in the family of high social status in Lycia. He studied rhetoric, philosophy and mathematics in Alexandria. At the age of 20 years, Proclus moved to Athens, where **Plutarch of Athens** was a head of Platonic Academy. As early of the age of 28 years, **Proclus** wrote one of his most important works, a commentary on Plato's *Timaeus.* About 450 he has became a head of the Platonic Academy.

The *Commentary on Book I of Euclid's Elements* is the most important Proclus' mathematical work. In his *Commentary*, **Proclus** put forward the following unusual hypothesis. As mentioned, the final book of the *Elements* (the Book XIII), is devoted to the presentation of the geometric theory of the five *regular* *polyhedra*, which play the leading role in *Plato's Cosmology* and is known in modern science as *Platonic solids* (Fig. 2). Typically, the most important material of scientific work takes a place in its final part. **Proclus** attracts a special attention to this fact. He believed that the placement of the theory of regular polyhedra in Book XIII is no accidental. Based on this observation, he put forth the following unusual hypothesis:

> **Proclus' hypothesis:** *By placing the theory of Platonic solids in the final Book (Book XIII) of his Elements, Euclid points out to the main goal of the Elements, which consists in the creation of geometric theory of Platonic solids, symbolizing the Universe Harmony in Plato's Cosmology.*

As it is pointed out by Eduard Soroko [16], according to Proclus, Euclid's goal was not to set forth geometry itself, but to build the complete theory of regular polyhedra ("Platonic solids"). This theory was outlined by Euclid in the XIII-th, the concluding book of the *Elements*. This fact in itself is an indirect confirmation of Proclus' hypothesis. Typically the most important scientific information is placed in the final part of a scientific work.

To attain this goal, **Euclid** included the necessary mathematical information into the *Elements*. The most curious fact is that in the Book II he introduced the *golden ratio*, used by him for the creation of the geometric theory of the *dodecahedron* (see Fig. 2). In *Plato's Cosmology*, the regular polyhedra have been associated with the Universe Harmony. This means that Euclid's *Elements* are based on the "harmonic ideas" of Pythagoras and Plato, that is, Euclid's *Elements* are historically the *first variant of the "Mathematics of Harmony"* [6]. This unexpected view on the *Elements* leads us to conclusions, which radically alter our ideas about the history of mathematics, starting from Euclid.

Proclus's hypothesis points out on the deep connection between *Pythagorean "Mathem's"* and Euclid's Elements because Euclid's *Elements* include in themselves all the *Pythagorean "Mathem's"*: *arithmetic*, *geometry*, *harmonics* and *spherics*.

As it is known, the famous Russian mathematician academician **Kolmogorov** in the book [26] has identified the "key" problems, which stimulated the development of mathematics at the stage of its origin. He considered two such problems: the *counting problem* and the *measurement problem.* The *counting problem* resulted in the creation of *natural numbers*, while the *measurement problem* (*"incommensurable segments"*) resulted in the creation of *irrational numbers*. Geometry historically follows from the *measurement problem.* As it is known, the name of *"geometry"* is from the ancient Greek *geo*-"earth", *-metron* "measurement". The natural and irrational numbers together with geometry are the foundation of *Classical Mathematics*, which underlies the *classical theoretical physics* and *computer science* (see Fig. 3).

The *harmony problem*, based on *Platonic solids* and the *golden ratio*, is the foundation for the *Mathematics of Harmony*, which underlies the "*golden" theoretical physics* and *"golden" computer science* (see Fig. 4).

According to *Proclus' hypothesis*, the *harmony problem*, connected with *Platonic solids* and the *golden ratio*, is the main idea of Euclid's *Elements*, in which the "harmonic ideas" by Pythagoras and Plato's were embodied.

As it follows from Proclus' hypothesis, Euclid's *Elements* can be regarded as historically the first attempt to create a mathematical work, devoted to the description of the *Platonic solids*, the main indicators of the Universe Harmony in Plato's Cosmology.

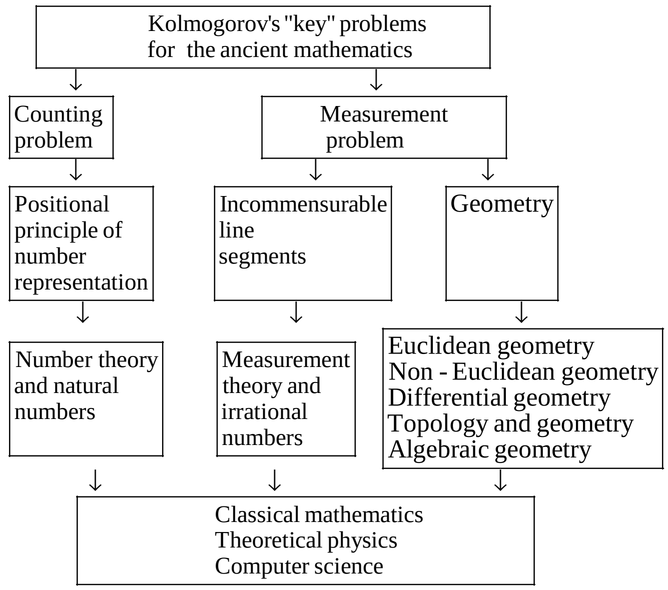


Fig. 3. Kolmogorov's way for the development of classical mathematics

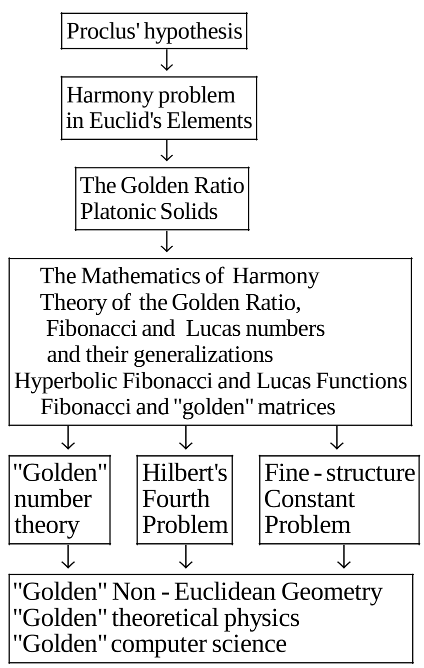


Fig. 4. Harmony mathematics and its applications following from Proclus' hypothesis

Theory of the *golden ratio, Fibonacci and Lucas numbers* [25,27,28] and their generalizations (*Fibonacci and Lucas λ-numbers* and *metallic means* [29-35]) are the basis of the Mathematics of Harmony. Hyperbolic Fibonacci and Lucas functions [12-15] and Fibonacci and "golden" matrices [36] follow from them.

The Mathematics of Harmony underlies such fundamental mathematical results as the "golden" number theory [21], original solutions to Hilbert's Fourth Problem as the MILLENNIUM PROBLEM in geometry and to Fine-Structure Constant Problem as Physical MILLENNIUN PROBLEM [22,23]. The above results underlie the *"Golden" Non-Euclidean Geometry, "Golden" Theoretical Physics, "Golden" Computer Science.*

This approach can be unexpected for many mathematicians. It turns out, that in parallel with *Classical Mathematics*, another mathematical discipline, the *Mathematics of Harmony*, begun to develop in mathematics since Euclid's *Elements,* which are the source for both directions.

However, the *Classical Mathematics* borrowed in Euclid's *Elements* the *axiomatic approach* and other ancient mathematical achievements (number theory, theory of irrationalities and so on), while the *Mathematics of Harmony* borrowed the *golden ratio* (Proposition II.11) and *Platonic Solids*, described in Book XIII of Euclid's *Elements*.

For many centuries, the creation of *Classical Mathematics* was the main goal of mathematicians. And in this direction, *Classical Mathematics* achieved outstanding successes and became the Queen of Natural Sciences.

Unfortunately, the fate of the *Mathematics of Harmony* was more dramatically. *Proclus' hypothesis* together with the *Mathematics of Harmony* was simply ignored in *Classical Mathematics*. The Mathematics of Harmony, as very important mathematical direction and (according to Proclus) the main idea of Euclid's *Elements*, developed very slowly.

However, it should be noted that the development of the Mathematics of Harmony, had never been stopped. Starting from **Pythagoras**, **Plato**, **Euclid** (ancient Greeks period), **Fibonacci** (the Middle Ages), **Pacioli, Kepler, Cassini** (the Renaissance), **Binet, Lucas, Klein** (19^{th} century), **Coxeter, Vorobyov, Hoggatt, Vajda** (20^{th} century), the intellectual forces of many prominent mathematicians and thinkers were directed towards the development of the basic concepts and applications of the *Mathematics of Harmony*. We have no right to ignore this important fact in the history of mathematics!

In the 19^{th} century, thanks to the efforts of French mathematicians **Binet** and **Lucas**, as well as the German mathematician **Felix Klein**, the new mathematical results in this area (*Binet's formulas* [37], *Lucas numbers* [38], *Lucas sequences* [39], *Klein's icosahedral idea* [40] and so on) were obtained.

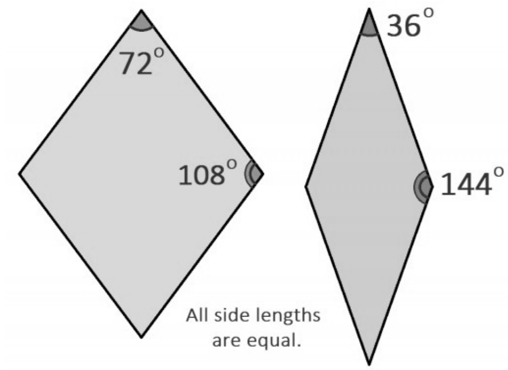
These mathematical achievements became a launching pad for the rapid development of this direction in the second half of the 20th century (the works of the Canadian geometer Harold Coxeter [41], the Soviet mathematician Nikolai Vorobyov [25], the American mathematician Verner Hoggatt [27], the English mathematician Stephan Vajda [28] and so on).

Unfortunately, starting since Euclid, these two important directions (*Classical Mathematics* and *Mathematics of Harmony*) evolved separately from one another. A time came to unite these important mathematical directions. This unusual union can lead to new scientific achievements in mathematics, computer science, theoretical natural sciences and become the source for future development of these sciences. The book [6] can become the basis for this union.

5 The Use of the Golden Ratio and Platonic Solids in Modern Science

5.1 "Parquet's problem" and Penrose's tiles

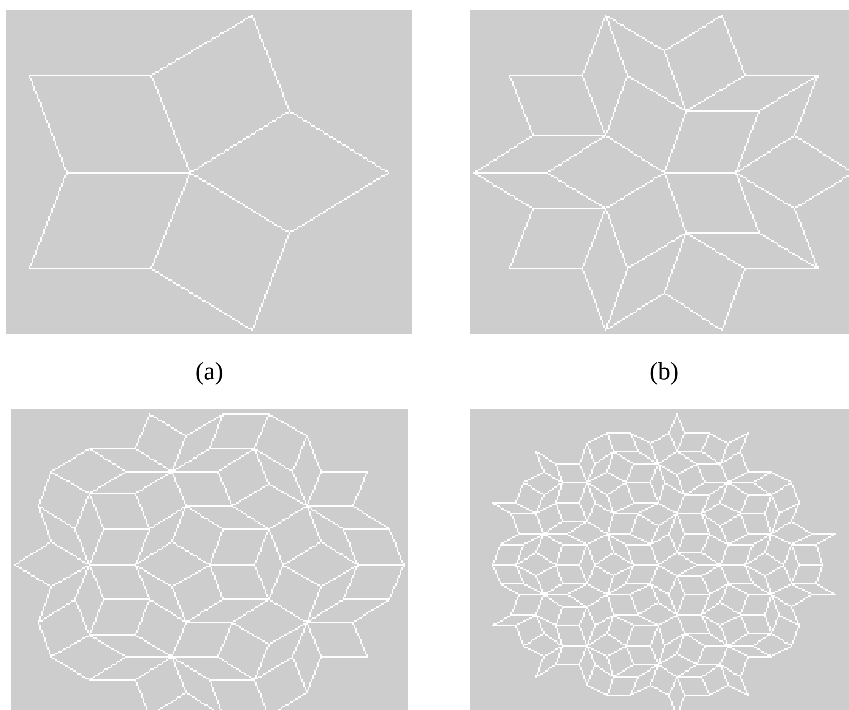
The English mathematician Sir **Roger Penrose** was the first researcher, who found an original solution of the *"parquet's problem"* known from ancient times. In 1972, he has covered a planar surface in non-periodic manner, by using only two simple polygons. In the simplest form, *Penrose's tiling* [42] is a non-random set of rhombi of two types, which follow directly from the regular pentagon and pentagram. The first one, called “*thick” rhombus* (Fig. 5a), has the internal angles 72° and 108° and the second one (Fig. 5b), called “*thin” rhombus*, has the internal angles 36° and 144°.



(a) **(b)**

Fig. 5. Penrose’s rhombi: “thick” rhombus (a) and “thin” rhombus (b)
(Wikipedia, the free encyclopedia, from https://www.google.ca/?gws_rd=ssl#q=penrose+rhombus)

As Sir **Roger Penrose** proved, the “thin” and “thick” rhombi in Fig. 5 allow covering completely an infinite planar surface. Below in Fig. 6, we can see a process of sequential constructing of *Penrose’s tiling* by using the *“thick”* and *“thin”* rhombi (Fig. 6).



(a) (b)

(c) (d)

Fig. 6. Penrose’s tiling
(Wikipedia, the free encyclopedia, from https://en.wikipedia.org/wiki/Penrose_tiling)

It is proved that the ratio of the number of the *"thick"* rhombi (Fig. 5a) to the number of the *"thin"* rhombi (Fig. 5b) for the *Penrose's tiling* (Fig. 6) strives to the *golden ratio* in the limit.

5.2 Quasi-crystals

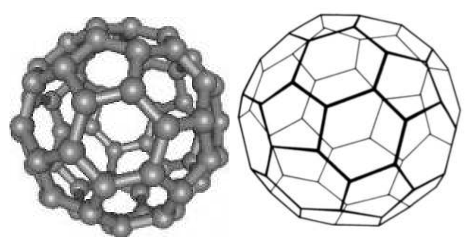
November 12, 1984, in the small article, published in the prestigious journal "Physical Review Letters" [43], the experimental proof of the existence of a metal alloy with exceptional properties has been presented. Israeli physicist **Dan Shechtman** was the author of this experimental discovery. This alloy has shown all indications of a crystal. Its diffraction pattern was made up of bright and regularly spaced points, just like a crystal. However, this picture has been characterized by the presence of *"icosahedral"* or *"pentagonal"* symmetry, strictly forbidden in the crystal from geometrical considerations. Such unusual alloys were called *quasi-crystals* [3].

Note that the *Penrose tiling* (Fig. 5) are planar model of the *quasi-crystals*.

As highlighted in Gratia's article [44], the notion of the *quasi-crystal "leads to expansion of crystallography, we only begin to explore the newly discovered wealth of quasi-crystals. Its importance in the world of minerals can be put on a par with the addition of the concept of irrational numbers to rational in mathematics"*.

5.3 Fullerenes

The *fullerenes* [4] are another outstanding scientific discovery, which has a relation to Platonic solids. This discovery was made in 1985 by **Robert F. Curl, Harold W. Kroto and Richard E. Smalley**. The title of *fullerenes* refers to the carbon molecules of the type C_{60}, C_{70}, C_{76}, C_{84}, in which all atoms are placed on a spherical or spheroid surface. In these molecules, the atoms of carbon are located at the vertexes of regular hexagons and pentagons that cover the surface of sphere or spheroid. The molecule of the carbon C_{60} (Fig. 7a), called *buckminsterfullerene,* plays a special role amongst fullerenes. It is based on the so-called *truncated icosahedron* [45] (Fig. 7b) and has the highest symmetry.



a) b)

Fig. 7. The molecule of the carbon C_{60} (a) and truncated icosahedron (b)
(Wikipedia, the free encyclopedia, from https://en.wikipedia.org/?title=Fullerene and https://en.wikipedia.org/wiki/Truncated_icosahedron)

Fullerenes (Nobel Prize in Chemistry-1996) and *quasi-crystals* (Nobel Prize in Chemistry-2011) are a worthy gift for the 100^{th} anniversary of the publication of the book [40], where **Felix Klein** after **Pythagoras**, **Plato** and **Euclid** predicted a prominent role of *Platonic solids* in modern science.

5.4 Fullerenes in the galaxy as the experimental confirmation of the self-organization and harmony of the universe

The article [46], published in the Journal *Nature,* contains sensational information. The article presents an experimental proof of the fact that the *Milky Way* and other galaxies contain a large amount of fullerenes.

Scientists say that fullerenes are actually contained in the galaxies in large amount what enshrines for them the status of *"nanofactories;"* so astronomers named them in 2011.

Commenting on this article, we can say that it contains very valuable information, which confirms the correctness of "harmonic ideas" by *Pythagoras, Plato, Euclid, Kepler, Klein,* who predicted the outstanding role of regular polyhedra ("Platonic solids") in the structures of science and Nature many millennia and centuries ago.

Through this experimental discovery, the *fullerenes*, based on the truncated icosahedron (Fig. 7b) and the *golden ratio,* acquired the status of the *main symbol of the "Universe Harmony."* Thanks to the Nobel Prizes for fullerenes (1996) and quasi-crystals (2011), theoretical natural sciences made great strides in the field of the "harmonic ideas" by Pythagoras and Plato.

6 A Discussion of Proclus' Hypothesis in Historical-mathematical Literature

6.1 The books by Charles H. Kann, Leonid Zhmud and Craig Smorinsky

The analysis of Proclus' hypothesis is found in many mathematical books. Consider some of them [47-49]. In **Charles H. Kann's** book [47], we read: *"According to Proclus, the main objective of the "Elements" was to present the geometric construction of the so-called Platonic solids."*

In **Leonid Zhmud's** book [48], this idea got a further development: *"Proclus, by mentioning all previous mathematicians of Plato's circle, said: "Euclid lived later than the mathematicians of Plato's circle, but earlier than Eratosthenes and Archimedes ... He belonged to Plato's school and was well acquainted with Plato's philosophy and his cosmology; that's why he put a creation of the geometric theory of the so-called Platonic solids as the main purpose of the Elements."*

This comment draws our attention to the deep connection between **Euclid** and **Plato**. Euclid fully shared Plato's philosophy and cosmology, based on Platonic solids, that is why, Euclid put forward the creation of the geometric theory of Platonic solids as the main purpose of his Elements.

In **Craig Smorinsky's** book [49], there is discussed the influence of Plato's and Euclid's ideas on **Johannes Kepler** at designing of the so-called *Kepler's Cosmic Cup* [50] (the original model of the Solar System, based on Platonic Solids in his first book "Mysterium Cosmographicum"):

> *"Kepler's project in "Mysterium Cosmographicum" was to give "true and perfect reasons for the numbers, quantities, and periodic motions of celestial orbits." The perfect reasons must be based on the simple mathematical principles, which had been discovered by Kepler in the Solar system, by using geometric demonstrations. The general scheme of his model was borrowed by Kepler from Plato's Timaeus, but the mathematical relations for the Platonic solids (pyramid, cube, octahedron, dodecahedron, icosahedron) were taken by Kepler from the works by Euclid and Ptolemy. Kepler followed Proclus and believed that "the main goal of Euclid was to build a geometric theory of the so-called Platonic solids." Kepler was fascinated by Proclus and often quotes him calling him "Pythagorean".*

From this quote, we can conclude that Kepler used the *Platonic solids* to create the *Cosmic Cup*, but all the mathematical relations for the Platonic solids were borrowed by him from Book XIII of the *Elements.* In other words, he united in his studies Plato's Cosmology with Euclid's *Elements.* Following this, he fully believed in *Proclus' hypothesis* that the main goal of Euclid was to create a complete geometric theory of the Platonic solids, which was used by Kepler in his geometric model of the Solar system.

6.2 Klein's icosahedronical idea

In the late of the 19th century, the German mathematician **Felix Klein** drew attention to the Platonic solids. He predicted an outstanding role of the Platonic solids, in particular, the icosahedron for the future development of science and mathematics. In 1884, Felix Klein published the book [40], dedicated to the geometric theory of the icosahedron.

Klein considers the regular icosahedron as the mathematical object, from which the branches of the five mathematical theories follow, namely, *geometry, Galois' theory, group theory, invariants theory and differential equations.*

Thus, **Felix Klein**, by following **Pythagoras**, **Plato**, **Euclid**, and **Johannes Kepler**, attracted attention to the fundamental role of the Platonic solids, in particular, the icosahedron, for the future development of science and mathematics. Klein's main idea is extremely simple [40]: *"Each unique geometrical object is somehow or other connected to the properties of the regular icosahedron".*

6.3 The opinion of the Russian historian of mathematics Mordukhai-Boltovskii

In comments to Euclid's *Elements* [51], Prof. D.D. Mordukhai-Boltovskii (the authoritative Russian historian of mathematics and translator of Euclid's *Elements* into Russian [51]) writes the following:

> *"After the careful analysis of Euclid's Elements, I have been convinced firmly that the construction of regular polyhedra, and even more - the proof of the existence of five and only five regular polyhedra - represented the ultimate goal of the work, which led to the origin of the Elements".*

Thus, many famous scientists, mathematicians and thinkers, starting from **Johannes Kepler** and **Felix Klein**, and ending by contemporary historians of mathematics **Dmitry Mordukhai-Boltovskii, Charles H. Kann, Leonid Zhmud and Craig Smorinsky,** firmly believed in the correctness of *Proclus' hypothesis*, which overturns our views on Euclid's Elements and mathematics history starting from Euclid. And we can not ignore these historical facts!

7 The "Golden" Number Theory and New Properties of Natural Numbers

7.1 Binary system, Bergman's system and codes of the golden *p*-proportions

A detailed description of the new approach to the creation of the "golden" number theory is given in the article [21].

We can use the binary system

$$A=\sum_i a_i 2^i \left(i=0,\pm1,\pm2,\pm3,\ldots\right) \tag{9}$$

for the constructive definition of real numbers *A*. Here $a_i \in \{0,1\}$ is a binary numeral, 2^i is the weight of the *i*-th digit, the number 2 is the base of the numeral system (9).

The definition (9) can be generalized by using the *codes of the golden p-proportions*, introduced in [10]:

$$A=\sum_i a_i \Phi_p^i;\ p=0,1,2,3,\ldots \tag{10}$$

where $a_i \in \{0,1\}$ is the binary numeral, Φ_p^i is the weight of the *i*-th digit; Φ_p is the base of the numeral system (10), the positive root of the following algebraic equation:

$$x^{p+1} - x^p - 1 = 0 , \tag{11}$$

where *p* is an integer number taken its value from the set $S \in \{0,1,2,3,...\}$.

Note that numeral systems (10) have the following properties:

1. A number of new positional binary ($a_i \in \{0,1\}$) representations of real numbers, given by (10), are theoretically infinite.
2. For the case *p*=0 the numeral systems (10) is reduced to the binary system (9).
3. The powers of the golden *p*-proportions Φ_p^i are connected by the following identities:

$$\Phi_p^i = \Phi_p^{i-1} + \Phi_p^{i-p-1}; \Phi_p^i = \Phi_p \times \Phi_p^{i-1} \left(i = 0, \pm1, \pm2, \pm3, ...\right). \tag{12}$$

4. Except for the case *p* = 0, all the rest golden *p*-proportions Φ_p (*p*=1,2,3,…) are irrational. This means that the formula (10) defines a new class of numeral systems, *numeral systems with irrational bases.*
5. For the case *p*=1 the numeral system (10) is reduced to *Bergman's system*, introduced in 1957 by 12-years American wunderkind **George Bergman** [52]:

$$A = \sum_i a_i \Phi^i \left(i = 0, \pm1, \pm2, \pm3, ...\right), \tag{13}$$

where *A* is a real number, $a_i \in \{0,1\}$ is a binary numeral, Φ^i is the weight of the *i*-th digit, $\Phi = \frac{1+\sqrt{5}}{2}$ (the *golden ratio*) is the base of Bergman's system (13).

It is necessary to note that the *number system with irrational base*, introduced by **George Bergman** in 1957 [52] and *codes of the golden p-proportions*, introduced and studied by the author in the 1984 book [10], are the most important mathematical discoveries of 20 c. in the field of numeral systems after the discovery of positional principle of number representation (Babylon, 2000 B.C.), decimal system (India, 5th century) and binary system (9). The most surprising is the fact that **George Bergman** made his mathematical discovery, having fundamental importance for number theory, in the age of 12 years! This is an unprecedented case in the history of mathematics!

7.2 New properties of natural numbers

The article [21] describes another unusual mathematical result: a discovery of new properties of natural numbers. The *Z-property* of natural numbers is one of the most surprising among them.

By using the concept of the $\Phi-$code of natural number *N*:

$$\Phi - code : N = \sum_i a_i \Phi^i \left(i = 0, \pm1, \pm2, \pm3, ...\right) \tag{14}$$

and the well-known formula

$$\Phi^i = \frac{L_i + F_i\sqrt{5}}{2}\left(i = 0, \pm1, \pm2, \pm3, \ldots\right), \quad (15)$$

which represents the golden ratio powers Φ^i through Lucas number L_i and Fibonacci numbers F_i, we can prove the following theorem.

Theorem (*Z*-property). If we represent an arbitrary natural number N in the Φ-code (14) and then substitute the Fibonacci number F_i instead of the golden ratio power Φ^i in the expression (14), where the index i takes its values from the set $\{0,\pm1,\pm2,\pm3,\ldots\}$, then the sum that appear as a result of such a substitution is equal to 0 identically, independently on the initial natural number N, that is,

$$\textit{For any } N = \sum_i a_i \Phi^i \textit{ after substitution } F_i \to \Phi^i : \sum_i a_i F_i \equiv 0 \left(i = 0, \pm1, \pm2, \pm3, \ldots\right) \quad (16)$$

Note that the *Z-property* and other unusual properties, described in [21], are valid only for natural numbers. It is surprising for many mathematicians to know that the new mathematical properties of natural numbers were discovered only at the beginning of the 21-th century, that is, 2½ millennia after the beginning of their theoretical study in Greek mathematics. The *golden ratio* and Fibonacci numbers play a fundamental role in this discovery. This discovery connects together two outstanding mathematical concepts of Greek mathematics, *natural numbers* and the *golden ratio.* This discovery is a confirmation of the fruitfulness of the constructive approach to the number theory based on *Bergman's system* (13) and *codes of the golden p-proportions* (10).

7.3 Applications in computer science

As it is shown in the book [6], *Bergman's system* (13) and *codes of the golden p-proportions* (10) have interesting applications in *computer science* and *digital metrology*. They are the basis for designing *Fibonacci computers* and *self-correcting analog-to-digit and digit-to-analog converters.*

8 Hilbert's Fourth Problem as a Candidate on the MILLENNIUM PROBLEM in Geometry

8.1 Hilbert's Fourth Problem as interdisciplinary problem

In the lecture *Mathematical Problems* [53], presented at the Second International Congress of Mathematicians (Paris, 1900), the prominent mathematician **David Hilbert** (1862-1943) formulated his famous 23 mathematical problems. Most of these problems have now been solved. Hilbert Fourth Problem is considered unsolved in spite of the attempts of the mathematicians (Hamel, Pogorelov) to solve this problem. In [53], this problem is formulated as follows:

> "*The more general question now arises: Whether from other suggestive standpoints geometries may not be devised which, with equal right, stand next to Euclidean geometry"*.

Hilbert's quote contains the formulation of very important mathematical problem, which has a multidisciplinary character, and, according to Hilbert, concerns to the foundation of *geometry, number theory, theory of surfaces* and *calculus.*

Wikipedia article [54] so formulates a status of Hilbert's Fourth Problem: *"the original statement of Hilbert has been judged too vague to admit a definitive answer"*.
This quote puts all blame for the solution to Hilbert's Fourth Problem (or rather the lack of the solution) on Hilbert himself, who formulated this problem *very vague*.

8.2 From the "game of postulates" to the "game of functions"

According to [55], the cause of the difficulties, arising at the solution of Hilbert's Fourth Problem, lies in the following. All the known attempts to solve this problem (Herbert Hamel, Alexey Pogorelov [56]) were based on the traditional approach and have been reduced to the so-called *"game of postulates"* [55]. This "game" in geometry started from the works by **Nikolai Lobachevski** and **Janos Bolyai**, when *Euclid's 5th postulate* was replaced on the opposite one. This was the most major step in the development of the *non-Euclidean geometry*. This changed the traditional geometric ideas and led to the creation of *hyperbolic geometry*. It must be emphasized that the title of *hyperbolic geometry* highlights the fact that this geometry is based on the *hyperbolic functions*. The use of hyperbolic functions for mathematical description of *hyperbolic geometry* was one of main Lobachevski's ideas.

8.3 New approach to the solution of Hilbert's fourth problem

Once again we emphasize that the very title of *hyperbolic geometry* contains in itself the important idea for another approach to the solution of Hilbert's Fourth Problem. This idea consists in *searching new classes of hyperbolic functions,* which can be the basis for new hyperbolic geometries. Every new class of the hyperbolic functions "generates" new variant of hyperbolic geometry. By analogy with the *game of postulates* this approach to the solution of Hilbert's Fourth Problem can be named the *game of functions* [55].

8.4 The recursive hyperbolic Fibonacci and Lucas functions based on the golden ratio

New classes of hyperbolic functions are introduced in [12-15]. Let us consider the simplest recursive hyperbolic functions based on the *golden ratio* $\Phi = \frac{1+\sqrt{5}}{2}$:

Hyperbolic Fibonacci sine:

$$sF(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}} \tag{17}$$

Hyperbolic Fibonacci cosine:

$$cF(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}} \tag{18}$$

Hyperbolic Lucas sine:

$$sL(x) = \Phi^x - \Phi^{-x} \tag{19}$$

Hyperbolic Lucas cosine:

$$cL(x) = \Phi^x + \Phi^{-x} \tag{20}$$

Hyperbolic functions (17) – (20) preserve all well-known properties of the classical hyperbolic functions, in particular *parity properties*:

$$\underline{\textbf{Parity property}}:$$
$$\begin{cases} sF(-x)=-sF(x); & cF(-x)=cF(x) \\ sL(-x)=-sL(x); & cL(-x)=cL(x) \end{cases} \quad (21)$$

The identity

$$\left[ch(x)\right]^2-\left[sh(x)\right]^2=1 \quad (22)$$

is possibly one of the most important properties of the classical hyperbolic functions. For the case of the hyperbolic functions (17) – (20) the identity (22) looks as follows:

$$\left[cF(x)\right]^2-\left[sF(x)\right]^2=\frac{4}{5} \quad (23)$$

$$\left[cL(x)\right]^2-\left[sL(x)\right]^2=4. \quad (24)$$

However, the *recursive properties* are the main feature of hyperbolic Fibonacci and Lucas functions (17) – (20). Here are some examples of *recursive properties*:

$$\begin{aligned} &\underline{\textbf{Recurrence relation for the Fibonacci hyperbolic functions}}: \\ &sF(x+2)=cF(x+1)+sF(x) \\ &cF(x+2)=sF(x+1)+cF(x) \\ &\underline{\textbf{Recurrence relation for the Lucas hyperbolic functions}}: \\ &sL(x+2)=cL(x+1)+sL(x) \\ &cL(x+2)=sL(x+1)+cL(x) \end{aligned} \quad (25)$$

$$\begin{aligned} &\underline{\textbf{Cassini's formula}}: \\ &\left[sF(x)\right]^2-cF(x+1)cF(x-1)=-1 \\ &\left[cF(x)\right]^2-sF(x+1)sF(x-1)=1 \end{aligned} \quad (26)$$

New hyperbolic geometry of phyllotaxis, based on the recursive hyperbolic Fibonacci functions (17), (18) (*Bodnar's geometry* [57]), is brilliant confirmation of practical usefulness of the functions (17) – (20) in Nature.

8.5 General theory of the recursive hyperbolic functions and original solution to Hilbert's fourth problem

The articles [15,22] describe a general theory of the recursive hyperbolic functions, based on the so-called *Fibonacci λ-numbers* and *"metallic proportions"* [29-35]. This theory underlies the original solution to Hilbert's Fourth Problem, which leads us to the "Golden" Non-Euclidean geometry and new challenge to theoretical natural sciences.

9 The Fine-structure Constant Problem as the Physical MILLENNIUM PROBLEM

The article [23] describes an original solution to the *fine-structure constant* problem based on the Mathematics of Harmony [6].

In 2000 a group of the eminent physicists has formulated the 10 Physics MILLENIUM PROBLEMS. These Physics MILLENIUM PROBLEMS have been presented at the *Strings 2000 Conference* (July, 10-15, University Michigan, Ann Arbor).

The first Physics MILLENIUM PROBLEM, formulated by David Gross (University of California, Santa Barbara), Nobel Prize Laureate in Physics-2004, sounds as follows:

> "*Are all the (measurable) dimensionless parameters that characterize the physical universe calculable in principle or are some merely determined by historical or quantum mechanical accident and incalculable?*"

The authors of the article [23] have focused on the *fine-structure constant (Sommerfeld's constant)*, which is a fundamental dimensionless physical constant, characterizing the strength of the electromagnetic interaction between elementary charged particles.

Bearing in mind the *fine-structure constant* as the main dimensionless physical constants of physical world, the authors of the article [23] have reformulated David Gross' MILLENNIUM PROBLEM as follows:

> *"Is the fine-structure constant, which characterizes the physical universe, calculable or non calculable?"*

Based on *Mathematics of Harmony* [6], the *"golden" matrices* [36] and *Fibonacci special theory of relativity* [58], the authors of the article [23] have deduced the mathematical formula that determines the dependence of the *fine-structure constant* from the time *T* since the *Big Bang*.

This formula makes it possible to calculate the values of the *fine-structure constant* for all stages of evolution of the Universe starting since the *Big Bang* (the *Dark Ages*, the *Light Ages*) and the *Black Hole* (the negative arrow of time).

It is proved in [23] a high coincidence of theoretical data for the value of the *fine structure constant* α with "the experimental data for the *Light Ages* of the Universe".

A substantiation of the coincidence between the theoretical and experimental data for the Black Hole and the Dark Ages is not possible. Such experimental data in physics and astronomy do not exist yet. However, there is given in [23] theoretical and numerical picture of the change of the fine-structure constant for the Black Hole, and for the Dark Ages.

10 Conclusions and Suggestions

Proclus hypothesis is extremely important and revolutionary idea for the history of mathematics. That is why, all mathematicians, math teacher, math students and representatives of theoretical natural sciences should be familiar with this amazing hypothesis, which overturns our understanding of Euclid *Elements* and mathematics history.

If we look at the origin of the term of "mathematics," we can find that this word unites four interconnected Pythagoras "Matem's": *arithmetic, geometry, harmonics* and *spherics.* From Pythagoras "Matem's," many mathematical directions and disciplines have been developed: algebra and calculus, mathematical analysis, probability theory, group theory, spherical and non-Euclidean geometry, formal logic, topology, and other mathematical discipline. Unfortunately, in the evolutionary process, one of the “Matem’s” was *lost*: at the certain stage of mathematics development; it was decided to *strike out* the *harmonics* from *Matem's* list.

Proclus Diadochus (412-485), the Neoplatonic philosopher and mathematician, one of the first commentators of Euclid’s Elements, was the first thinker, who has tried to correct this *strategic mistake* in mathematics history. *Proclus hypothesis* is one of the most unusual hypotheses in mathematics history. According to *Proclus hypothesis,* Euclid’s *Elements* are an embodiment of “harmonic ideas” by Pythagoras and Plato and they can be treated as historically the first version of the *Mathematics of Harmony*, one of the most important directions in mathematics development.

Note that *harmonic ideas* by Pythagoras, Plato and Euclid are widely used in theoretical natural sciences, particularly in modern theoretical physics, chemistry, crystallography, botany, biology, medicine and so on. This is confirmed by a number of outstanding scientific discoveries honoured with the Nobel Prize (fullerenes, quasi-crystals and others).

The following conclusions and suggestions follow from these arguments:

1. Historical truth must prevail: Proclus hypothesis should be introduced into modern mathematics and especially into mathematical education.
2. Introduction of Proclus hypothesis into modern mathematics can lead to the revision of the *strategic mistake* in the development of mathematics, which led to the exclusion of one of the Pythagorean Mathem’s (*harmonics*) from the foundation of mathematics.
3. Introduction of Proclus hypothesis into mathematics can lead to increasing the interest in the Mathematics of Harmony [6] as a new interdisciplinary direction of modern science. This can lead to the wider use of the *golden ratio, Fibonacci numbers* and *Platonic solids* in theoretical natural sciences what is a prerequisite for new scientific discoveries.
4. Introduction of Proclus hypothesis into mathematics can lead to enhancing the role of the *golden ratio*, *Fibonacci numbers* and *Platonic solids* in mathematical education. These mathematical and geometrical concepts should play the same role in mathematical education as for example, *Pythagorean Theorem.*

Competing Interests

Author has declared that no competing interests exist.

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