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Equidistant Set of Two Congruent Spheres and Its Orthogonal Projection on $\mathbb{R}^k$

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this paper some properties of equidistant sets are presented,a relatively new concept. The equidistant concept is characterized and among two congruent spheres of $\mathbb{R}^n$. Afterwards the behavior of the orthogonal projection onto $\mathbb{R}^k$ is studied, concluding that the projection of equidistant set of two congruent spheres, is a translation of the equidistant set of the spheres projections.

Keywords: Midset; orthogonal projection; spheres; metric.

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1 Introduction

The origin of equidistant concept is not clear. Leibniz in 1849 suggested defining a plane as the locus of points equidistant from two given points [1]. Busemann in [1], introduces the first formal definition of equidistant sets, using the name "bisector". Berard in [2] introduces the concept of "midset" to identify the equidistant sets, their properties and find sufficient conditions to make

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them homeomorphic to intervals of real numbers and the relationship between connected and midset. Loveland in [3, 4, 5] consolidates a line of research around the concept of midset, however on singleton sets and presents the first conjecture around the concept of midset. Wilker in [6], generalizes the concept of equidistant for any sets based on the infimum of the distances. Loveland [7, 8] continuing with studies and proves his conjecture.

On the other hand, Nadler in [9] shows some relationships between equidistant sets and the real line. Later [10, 11, 12] proposes a new conjecture and incorporating a generalization of the double midset property.

Debski in [13] solves the problem that If X be a non-degenerate metric space such that each of its midsets is homomorphic to an (n-1)-sphere, then is X homomorphic to an n-sphere. [14] introduces the concept of metric space X (m,n)-equidistant and a relation with the conjecture of [4].

Finally Ponce and Santibañez [15], proposed the midset as generalized conics, they also propose an error estimate result about approximative version of equidistant sets.

This proposal is based on characterizing the set equidistant among two congruent spheres of $\mathbb{R}^n$. To further, study the behavior of of the orthogonal projection onto $\mathbb{R}^k$ is studied, concluding that the projection of equidistant set of two congruent spheres, is a translation of the equidistant set of the spheres projections.

This paper is organized as it follows: in section 1 the definition of equidistant set is formalized. In section 2, topological properties of equidistant sets are presented. In Section 3, the equidistant set of two congruent spheres is determined and finally, we will study the behavior of the orthogonal projection on $\mathbb{R}^k$.

1.1 Preliminaries

The organization of the following definitions is presented so as this article is self- contained, for this reason we describe some basic concepts for the understanding of our work.

Definition 1.1. Let (X, d^*) be a metric space and $A, B \subseteq X$ not empty. The distance between A and B is defined

$$\begin{array}{rlcl} \tilde{d_1}: & (\mathcal{P}(X) \setminus \phi) \times (\mathcal{P}(X) \setminus \phi) & \longrightarrow & \mathbb{R} \\ & (A, B) & \longmapsto & \tilde{d_1}(A, B) = \inf_{\substack{b \in B \\ a \in A}} \{d^*(a, b)\} \end{array}$$

Where $(\mathcal{P}(X) \setminus \phi)$ is the set of parts X without the empty set.

Remark 1.1. As a particular situation from Definition 1.1 we can see that

$$\tilde{d_1}(A, B) = \inf_{b \in B} \{d^*(a, b)\}$$

if $A = \{a\}$.

Formalizing the Observation 1.1 we have:

Definition 1.2. Let $x \in X$ be and $A \subseteq X$ not empty set, then the distance between x and A is:

$$\tilde{d}(x, A) = \inf_{a \in A} \{d^*(x, a)\}.$$

Based on the definitions 1.2 and 1.1, we establish the fundamental concept of this work, the definition of equidistant set among two non empty sets:

Definition 1.3. Let A and B be two non-empty set, the set equidistant between A and B is given by

$$\{A = B\} := \left\{x \in X : \tilde{d}(x, A) = \tilde{d}(x, B)\right\},$$

where (X, d^*) is a metric space.

Next, we will use the notation $\{A = B\}$ which is used to indicate the equidistant set between A and B, this notation was introduced by Wilker [6]. Similarly, Loveland [3] introduced the term *midset* to refer to the same set.

2 Some Situations on $\mathbb{R}$ and Examples on $\mathbb{R}^2$

In this section we will analyze the different situations of equidistant sets on $\mathbb{R}$ and we will show examples on $\mathbb{R}^2$.

2.1 Situations on $\mathbb{R}$

Given $A, B \subseteq \mathbb{R}$ non-empty sets, we have:

1. If $A = \{a\}$ and $B = \{b\}$ with $a \neq b$, then

$$\{A = B\} = \left\{\frac{a+b}{2}\right\}.$$

2. If $A = [a, b]$ and $B = [c, d]$, we have the following cases:

   - $A \cap B = \emptyset$, then, without loss of generality we can consider that $a < b < c < d$, then
   $$\{A = B\} = \left\{\frac{b+c}{2}\right\}.$$
   - $A \subseteq B$, then
   $$\{A = B\} = A.$$
   - $A \cap B \neq \emptyset$, $A \not\subseteq B$ and $B \not\subseteq A$, then
   $$\{A = B\} = A \cap B.$$

3. If $A = \{a\}$ and $B = [c, d]$, we have the following cases:

   - $A \subseteq B$, then
   $$\{A = B\} = A.$$
   - $A \cap B = \emptyset$, without loss of generality, let us consider $a < c < d$, then
   $$\{A = B\} = \left\{\frac{a+c}{2}\right\}.$$

4. Let us consider $A = \{\sqrt{p}\}$ where p is a prime number and $B = \mathbb{Q}$, then as $\sqrt{p} \in \overline{\mathbb{Q}}$, we have $\{\sqrt{p}\} \subseteq \overline{\mathbb{Q}}$, then by Proposition 3.1 we obtain

$$\{\{\sqrt{p}\} = \mathbb{Q}\} = \{\overline{\{\sqrt{p}\}} = \overline{\mathbb{Q}}\} = \{\sqrt{p}\},$$

given that $\{\sqrt{p}\} \cap \overline{\mathbb{Q}} = \{\sqrt{p}\}$.

2.2 Examples on $\mathbb{R}^2$

Example 2.1. *Let us consider the sets*

$$A = \{(x, y) \in \mathbb{R}^2 : 2 \leq y \leq 4 \wedge 0 \leq x \leq 2\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 2 \wedge 4 \leq x \leq 6\},$$

then the equidistant set between A and B is (see Fig.1.)

$$\{A = B\} = \begin{cases} 220x - 111y = 334 & \text{if } (x, y) \in (-\infty, 0] \times (-\infty, -3] \\ 4y + x^2 - 8x = -12 & \text{if } (x, y) \in [0, 2] \times [-3, 0] \\ y + 2\sqrt{3 - x} = 2 & \text{if } (x, y) \in [2, 3] \times [0, 2] \\ y - 2\sqrt{x - 3} = 2 & \text{if } (x, y) \in [3, 4] \times [2, 4] \\ -4y + x^2 - 4x = -16 & \text{if } (x, y) \in [4, 6] \times [4, 7] \\ -11xx + 56y = 278 & \text{if } (x, y) \in [6, \infty) \times [7, \infty) \end{cases}$$

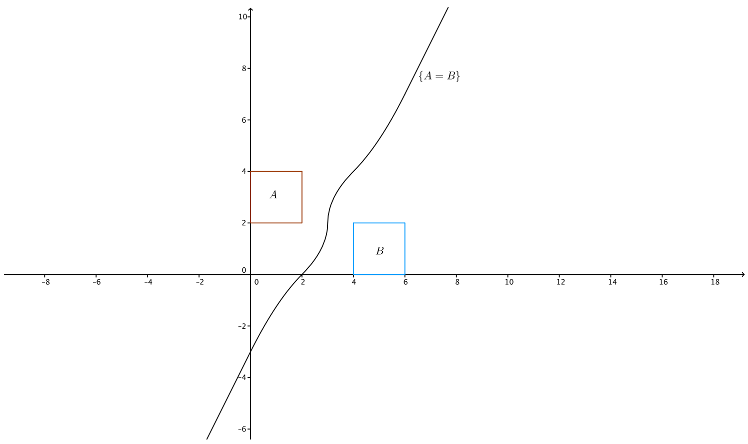

Fig. 1. Graphic of the equidistant set Example 2.1

Example 2.2. *Let us consider the sets*

$$A = \{(x, y) \in \mathbb{R}^2 : 2 \leq y \leq 4 \wedge 0 \leq x \leq 2\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 2 \wedge 2 \leq x \leq 6\},$$

then the equidistant set between A and B is (see Fig. 2)

$$\{A = B\} = \begin{cases} x - y = 0 & \text{if } (x, y) \in (-\infty, 4] \times (-\infty, 4] \\ x^2 - 4x - 4y = -16 & \text{if } (x, y) \in [4, 6] \times [4, 7] \\ -241x + 120y = 602 & \text{if } (x, y) \in [6, \infty) \times [7, \infty) \end{cases}$$

We note that the equidistant set between these sets that define polygons, are defined by a piecewise relation defined by parabola, bisector, perpendicular bisector and straights.

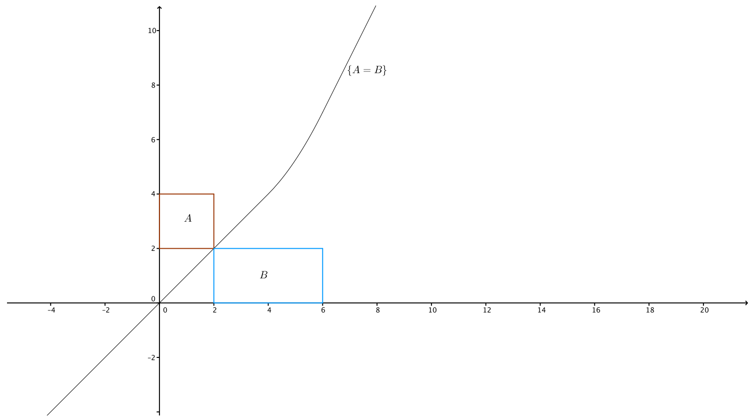

Fig. 2. Graphic of the equidistant set from Example 2.2

3 Properties of Equidistant Sets

The properties of equidistant sets that are presented below, are defined on any metric space.

Proposition 3.1. *Let (X, d^*) be a metric space and $A, B \subseteq X$ non-empty sets, then the following properties are held:*

1. $\{A = B\} = \{\overline{A} = \overline{B}\}$.
2. $\overline{A} \cap \overline{B} \subseteq \{A = B\}$.
3. *If* $\overline{A} \cup \overline{B} = X$ *then* $\{A = B\} = \overline{A} \cap \overline{B}$.
4. *If* $\overline{A} = \overline{B}$ *then* $\{A = B\} = X$.

Proof. 1. Note that $\overline{A} = \{x \in X : \tilde{d}(x, A) = 0\}$, then $\tilde{d}(x, A) = \tilde{d}(x, \overline{A})$ and $\tilde{d}(x, B) = \tilde{d}(x, \overline{B})$, therefore $\{A = B\} = \{\overline{A} = \overline{B}\}$.

2. Let be $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$, wich implies that $\tilde{d}(x, \overline{A}) = 0$ and $\tilde{d}(x, \overline{B}) = 0$. By previous item we have $\tilde{d}(x, A) = \tilde{d}(x, \overline{A})$ and $\tilde{d}(x, B) = \tilde{d}(x, \overline{B})$, then $\tilde{d}(x, A) = 0$ and $\tilde{d}(x, B) = 0$, thus $\tilde{d}(x, A) = \tilde{d}(x, B)$. Therefore $x \in \{A = B\}$.

3. We know that $\overline{A} \cap \overline{B} \subseteq \{A = B\}$, so it is only necessary to show that $\{A = B\} \subseteq \overline{A} \cap \overline{B}$, when $\overline{A} \cup \overline{B} = X$.

   Let $x \in \{A = B\}$ be then, $\tilde{d}(x, A) = \tilde{d}(x, B)$. By Item 1 $\tilde{d}(x, \overline{A}) = \tilde{d}(x, \overline{B})$, furthermore by hypothesis $x \in \overline{A}$ or $x \in \overline{B}$, implying that

$$\tilde{d}(x, \overline{A}) = 0 = \tilde{d}(x, \overline{B}),$$

   therefore $x \in \overline{A}$ and $x \in \overline{B}$, thus $x \in \overline{A} \cap \overline{B}$.

4. We trivially know that $\{A = B\} \subseteq X$, then we only need to show that $X \subseteq \{A = B\}$. Let $x \in X$ be, then we have the following possibilities:

   a) That $x \in A \cap B$, then $\tilde{d}(x, A) = \tilde{d}(x, B) = 0$, then $x \in \{A = B\}$, satisfies the thesis.

   b) That $x \in A$ and $x \notin B$, which is a contradiction, given that $x \in \overline{A} = \overline{B}$.

   c) That $x \notin A$ and $x \in B$, which is a contradiction, given that $x \in \overline{A} = \overline{B}$.

d) That $x \notin A \cap B$. We know that $A \cap B \subseteq \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. This tells us that $x \notin \overline{A}$ and $x \notin \overline{B}$, thus, we can conclude that $\tilde{d}(x, \overline{A}) = \tilde{d}(x, \overline{B})$ given that $\overline{A} = \overline{B}$. Which is equivalent to $\tilde{d}(x, A) = \tilde{d}(x, B)$, that is, $x \in \{A = B\}$.

□

4 Equidistant Set of Two Spheres $n - 1$ Dimensional

This section aims to describe the equidistant set of two congruent spheres $(n-1) - dimensional$, for this, the following definitions are necessary.

Definition 4.1. A sphere of radius r and center $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ is defined as

$$S_{r,c}^{n-1} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i - c_i)^2 = r^2 \right\}$$

Definition 4.2. A disc of radius r and center $c = (c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ is defined as

$$D_{r,c}^{n-1} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} (x_i - c_i)^2 < r^2 \right\}$$

Based on definitions 4.1 and 4.2 we have to:

Theorem 4.1. *Let $S_{r,e}^{n-1}$ and $S_{r,b}^{n-1}$ be two spheres and Γ a traslation such that,*

$$\Gamma\left(S_{r,e}^{n-1}\right) = S_{r,O}^{n-1}, \quad \text{with } O = (0, \ldots, 0) \in \mathbb{R}^n$$

and

$$\Gamma\left(S_{r,b}^{n-1}\right) = S_{r,a}^{n-1}, \quad \text{with } a = (a_1, \ldots, a_n) \in \mathbb{R}^n$$

Then

$$\left\{S_{r,O}^{n-1} = S_{r,a}^{n-1}\right\} = \left\{ x \in \mathbb{R}^n : x_n = -\sum_{i=1}^{n-1} \frac{a_i}{a_n} x_i + \frac{\|a\|^2}{2a_n} \right\},$$

with $x = (x_1, x_2, \ldots, x_n)$.

Proof. Let $S_{r,e}^{n-1}$ and $S_{r,b}^{n-1}$ be two spheres and Γ a translation with the above conditions. Then

$$\begin{aligned}
\left\{S_{r,O}^{n-1} = S_{r,a}^{n-1}\right\} &= \{x \in \mathbb{R}^n : \|x\| - r = \|x - a\| - r\} \\
&= \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (x_i - a_i)^2 \right\} \\
&= \left\{ x \in \mathbb{R}^n : 0 = -2 \sum_{i=1}^{n} x_i a_i + \|a\|^2 \right\} \\
&= \left\{ x \in \mathbb{R}^n : 0 = -2 \left(\sum_{i=1}^{n-1} x_i a_i + x_n a_n \right) + \|a\|^2 \right\} \\
&= \left\{ x \in \mathbb{R}^n : x_n = -\sum_{i=1}^{n-1} \frac{a_i}{a_n} x_i + \frac{\|a\|^2}{2a_n} \right\}
\end{aligned}$$

□

Remark 4.1. Midset is calculated considering a sphere in the origen to simplify the calculation. Then $\left\{S_{r,e}^{n-1} = S_{r,b}^{n-1}\right\} = \Gamma\left(\left\{S_{r,O}^{n-1} = S_{r,a}^{n-1}\right\}\right)$. Here $\Gamma = \Gamma_{\vec{e}}$.

Example 4.2. *Let $S^2_{1,e}$ and $S^2_{1,b}$ be two spheres such that $e = (1, 1, 1)$ and $b = (2, -1 - 1)$. Here $\Gamma = \Gamma_{-\vec{e}}$ such that:*

$$\Gamma\left(S^2_{1,e}\right) = \left\{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}$$

and

$$\Gamma\left(S^2_{1,b}\right) = \left\{x \in \mathbb{R}^3 : (x_1 - 1)^2 + (x_2 + 2)^2 + (x_3 + 2)^2 = 1\right\},$$

with $a = (1, -2, -2)$.
Then:

$$\left\{\Gamma\left(S^2_{1,e}\right) = \Gamma\left(S^2_{1,b}\right)\right\} = \left\{x \in \mathbb{R}^3 : x_3 = \frac{1}{2}x_1 + x_2 - \frac{9}{4}\right\}.$$

The equidistant set is illustrated in Fig. 3. Then:

$$\left\{S^2_{1,e} = S^2_{1,b}\right\} = \Gamma\left(\left\{\Gamma\left(S^2_{1,e}\right) = \Gamma\left(S^2_{1,b}\right)\right\}\right).$$

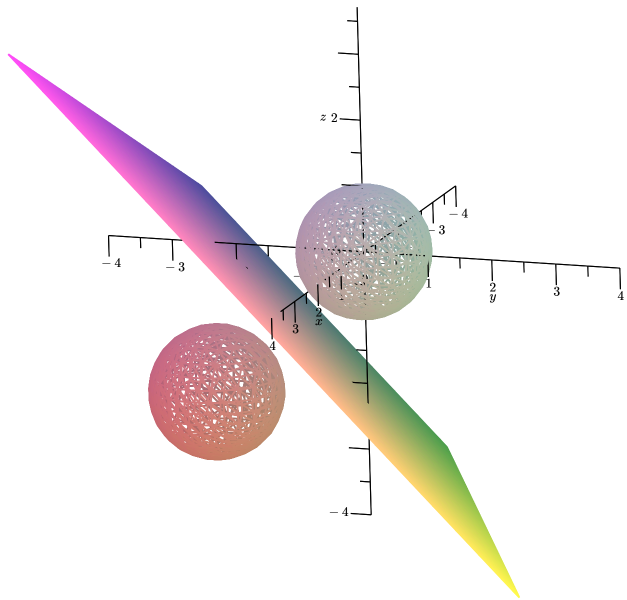

Fig. 3. $\left\{\Gamma\left(S^2_{1,e}\right) = \Gamma\left(S^2_{1,b}\right)\right\}$

5 Orthogonal Projection of $\mathbb{R}^n$ on $\mathbb{R}^k$

Once we determined the equidistant set among two congruent spheres, our purpose will be to study the behavior of the orthogonal projection onto $\mathbb{R}^k$.
For this, let's consider the following function:

$$\psi_{n,k} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that

$$\psi_{n,k}(x) = (x_1, \ldots, x_k, 0, \ldots, 0)$$

Note that the function $\psi_{n,k}$ is a projection of space $\mathbb{R}^n$ on the sub space $\mathbb{R}^k$, moreover, $\psi_{n,k}\left(S^{n-1}_{r,a}\right) = \overline{D^{k-1}}_{r,\psi_{n,k}(a)}$.

Theorem 5.1. *Let $S^{n-1}_{r,O}$ with $O = (0, \dots, 0) \in \mathbb{R}^n$ and $S^{n-1}_{r,a}$ with $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ be two spheres such that $a_n \neq 0$, $a_k \neq 0$. Then $\psi_{n,k}\left(\left\{S^{n-1}_{r,O} = S^{n-1}_{r,a}\right\}\right)$ corresponds to a translation of the equidistant set, according to vector* $\left(0, \dots, 0, \dfrac{\sum_{i=k+1}^{n} a_i^2}{2a_k}\right) \in \mathbb{R}^k$ *of the discs $D^{k-1}_{r,\psi_{n,k}(O)}$ and $D^{k-1}_{r,\psi_{n,k}(a)}$.*

Proof. Let $S^{n-1}_{r,O}$ with $O = (0, \dots, 0) \in \mathbb{R}^n$ and $S^{n-1}_{r,a}$ with $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ be two spheres. Then:

$$\begin{aligned}
\psi_{n,k}\left(\left\{S^{n-1}_{r,O} = S^{n-1}_{r,a}\right\}\right) &= \psi_{n,k}\left(\left\{x \in \mathbb{R}^n : x_n = -\sum_{i=1}^{n-1} \frac{a_i}{a_n} x_i + \frac{\|a\|^2}{2a_n}\right\}\right) \\
&= \left\{x \in \mathbb{R}^k : x_k = -\sum_{i=1}^{k-1} \frac{a_i}{a_k} x_i + \frac{\|a\|^2}{2a_k}\right\} \\
&= \left\{x \in \mathbb{R}^k : x_k = -\sum_{i=1}^{k-1} \frac{a_i}{a_k} x_i + \sum_{i=1}^{k} \frac{a_i^2}{2a_k} + \sum_{i=k+1}^{n} \frac{a_i^2}{2a_k}\right\} \\
&= \Gamma_\sigma\left(\left\{\psi_{n,k}\left(S^{n-1}_{r,O}\right) = \psi_{n,k}\left(S^{n-1}_{r,a}\right)\right\}\right),
\end{aligned}$$

where $\sigma = \dfrac{\sum_{i=k+1}^{n} a_i^2}{2a_k}$. □

Remark 5.1. As the equidistant above is a hyperplane, we can associate a function f to that hyperplane, as it follows:

$f : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that $f(x_1, \dots, x_{n-1}) = -\sum_{i=1}^{n-1} \dfrac{a_i}{a_n} x_i + \dfrac{\|a\|^2}{2a_n}$.

so we obtain:

$$\left\{\psi_{n,k}\left(S^{n-1}_{r,O}\right) = \psi_{n,k}\left(S^{n-1}_{r,a}\right)\right\} = \left\{x \in \mathbb{R}^{k-1} : f(x) = -\sum_{i=1}^{k-1} \frac{a_i}{a_k} x_i + \frac{\sum_{i=1}^{k} a_i^2}{2a_k}\right\}.$$

6 Conclusion

In this paper equidistant set examples about $\mathbb{R}$ y $\mathbb{R}^2$ were presented. As well as, their toplogical properties were shown. The midset of two spheres on $\mathbb{R}^n$ was characterized through a theorem. Together with this, its behavior of orthogonal projection on $\mathbb{R}^k$ was described.

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Competing Interests

The authors declare that no competing interests exist.

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