British Journal of Mathematics & Computer Science
10(2): 1-34, 2015, Article no.BJMCS.18442
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Fourier Coefficients of a Class of Eta Quotients of Weight 12

Barış Kendirli[1*]

[1]*Civil Engineering Department, Istanbul Kultur University, Turkey.*

Article Information

DOI: 10.9734/BJMCS/2015/18442
Editor(s):
(1) Nikolaos Dimitriou Bagis, Department of Informatics and Mathematics, Aristotelian University of Thessaloniki, Greece.
Reviewers:
(1) Anonymous, Kindai University, Japan.
(2) Anonymous, Norfolk State University, USA.
(3) Anonymous, The Pennsylvania State University, USA.
Complete Peer review History: http://sciencedomain.org/review-history/9950

Original Research Article

Received: 21 April 2015
Accepted: 02 June 2015
Published: 29 June 2015

Abstract

Recently, Williams [1] and then Yao, Xia and Jin [2] discovered explicit formulas for the coefficients of the Fourier series expansions of a class of eta quotients. Williams expressed all coefficients of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$ and Yao, Xia and Jin, following the method of proof of Williams, expressed only even coefficients of 104 eta quotients in terms of $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ and $\sigma_3\left(\frac{n}{6}\right)$. Here, we will express the odd Fourier coefficients of 334 eta quotients in terms of $\sigma_{11}(2n-1)$ and $\sigma_{11}\left(\frac{2n-1}{3}\right)$, i.e., the Fourier coefficients of the difference, f(q)-f(-q), of 334 eta quotients and we will express the even Fourier coefficients of 198 eta quotients i.e., the Fourier coefficients of the sum, f(q)+f(-q), of 198 eta quotients in terms of $\sigma_{11}(n), \sigma_{11}\left(\frac{n}{2}\right), \sigma_{11}\left(\frac{n}{3}\right), \sigma_{11}\left(\frac{n}{4}\right), \sigma_{11}\left(\frac{n}{6}\right)$ and $\sigma_{11}\left(\frac{n}{12}\right)$.

Keywords: Dedekind eta function; eta quotients; Fourier series.

1 Introduction

The divisor function $\sigma_i(n)$ is defined for a positive integer i by

$$\sigma_i(n) := \sum_{d \text{ positive integer}, d|n} d^i \text{, if } n \text{ is a positive integer, and} \tag{1}$$

$$\sigma_i(n) := 0 \text{ if n is not a positive integer.}$$

*Corresponding author: baris.kendirli@gmail.com;

The Dedekind eta function is defined by

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{2}$$

where

$$q := e^{2\pi iz}, z \in H = \{x+iy: y>0\} \tag{3}$$

and an eta quotient of level n is defined by

$$f(z) := \prod_{m|n}\eta\,(mz)^{a_m}, n \in \mathbb{N}, a_m \in Z, a_n \neq 0. \tag{4}$$

It is interesting and important to determine explicit formulas of the Fourier coefficients of eta quotients since they are the building blocks of modular forms of level n and weight k. The book of Köhler [3] (Chapter 3, pg. 39) describes such expansions by means of Hecke Theta series and develops algorithms for the determination of suitable eta quotients. One can find more information in [4,5,6,7,8]. I have determined the Fourier coefficients of the theta series associated to some quadratic forms, see [9,10,11,12,13,14].

Recently, Williams, see [1] discovered explicit formulas for the coefficients of Fourier series expansions of a class of 126 eta quotients in terms of $\sigma(n), \sigma(\frac{n}{2}), \sigma(\frac{n}{3})$ and $\sigma(\frac{n}{6})$. One example is as follows:

$$\frac{\eta^2(2z)\eta^4(4z)\eta^6(6z)}{\eta^2(z)\eta^2(3z)\eta^4(12z)}$$

gives the expansion found by Williams.

Then Yao, Xia and Jin [2] *expressed the even Fourier coefficients of 104 eta quotients in terms of* $\sigma_3(n), \sigma_3(\frac{n}{2}), \sigma_3(\frac{n}{3})$ *and* $\sigma_3(\frac{n}{6})$*.One example is as follows:*

$$\frac{\eta^{25}(2z)\eta^4(3z)}{\eta^{12}(z)\eta^5(4z)\eta^3(6z)\eta(12z)},$$

where the even coefficients are obtained. Motivated by these two results, we find that we can express the odd Fourier coefficients of 198 eta quotients in terms of $\sigma_{11}(2n+1)$ *and* $\sigma_{11}(\frac{2n+1}{3})$*, see Table 1, Supplementary.*

One example is as follows:

$$\eta^5(2z)\eta^4(4z)\eta^{13}(6z)\eta(12z),$$

where the odd coefficients of this eta quotient are obtained. We can also express the even Fourier coefficients of 334 eta quotients in terms of

$$\sigma_{11}(n), \sigma_{11}\left(\frac{n}{2}\right), \sigma_{11}\left(\frac{n}{3}\right), \sigma_{11}\left(\frac{n}{4}\right), \sigma_{11}\left(\frac{n}{6}\right) \text{ and } \sigma_{11}\left(\frac{n}{12}\right),$$

(See Table 2, Supplementary). One example is as follows:

$$\frac{\eta^{14}(6z)\eta^{14}(12z)}{\eta^2(2z)\eta^2(4z)}.$$

Let

$$f_1 := \sum_{n=0}^{\infty} f_1(n)q^n = \frac{\eta^{20}(4z)\eta^{10}(6z)\eta^4(12z)}{\eta^{10}(2z)}$$

$$f_2 := \sum_{n=0}^{\infty} f_2(n)q^n = \frac{\eta^{15}(4z)\eta^{15}(6z)\eta^3(12z)}{\eta^9(2z)}$$

$$f_3 := \sum_{n=0}^{\infty} f_3(n)q^n = \frac{\eta^{10}(4z)\eta^{20}(6z)\eta^2(12z)}{\eta^8(2z)}$$

$$f_4 := \sum_{n=0}^{\infty} f_4(n)q^n = \eta^5(2z)\eta^5(4z)\eta^{13}(6z)\eta(12z)$$

$$f_5 := \sum_{n=0}^{\infty} f_5(n)q^n = \frac{\eta^{16}(4z)\eta^2(6z)\eta^8(12z)}{\eta^2(2z)}$$

$$f_6 := \sum_{n=0}^{\infty} f_6(n)q^n = \frac{\eta^9(2z)\eta^9(6z)\eta^9(12z)}{\eta^3(4z)}$$

$$f_7 := \sum_{n=0}^{\infty} f_7(n)q^n = \frac{\eta^{11}(2z)\eta^{11}(4z)\eta^7(12z)}{\eta^5(6z)}$$

$$f_8 := \sum_{n=0}^{\infty} f_8(n)q^n = \frac{\eta^{12}(6z)\eta^{18}(12z)}{\eta^6(4z)}$$

$$f_9 := \sum_{n=0}^{\infty} f_9(n)q^n = \frac{\eta^{14}(2z)\eta^8(4z)\eta^{16}(12z)}{\eta^{14}(6z)}$$

$$f_{10} := \sum_{n=0}^{\infty} f_{10}(n)q^n = \frac{\eta^6(2z)\eta^{12}(4z)\eta^{18}(6z)}{\eta^{12}(12z)}$$

$$f_{11} := \sum_{n=0}^{\infty} f_{11}(n)q^n = \frac{\eta^{20}(2z)\eta^4(6z)\eta^4(12z)}{\eta^4(4z)}$$

$$f_{12} := \sum_{n=0}^{\infty} f_{12}(n)q^n = \frac{\eta^{19}(2z)\eta(4z)\eta^5(12z)}{\eta(6z)}$$

$$f_{13} := \sum_{n=0}^{\infty} f_{13}(n)q^n = \frac{\eta^{20}(4z)\eta^4(6z)\eta^4(12z)}{\eta^4(2z)}$$

$$f_{14} := \sum_{n=0}^{\infty} f_{14}(n)q^n = \frac{\eta^{14}(6z)\eta^{14}(12z)}{\eta^2(2z)\eta^2(4z)}$$

$$f_{15}:=\sum_{n=0}^{\infty} f_{15}(n)q^n=\frac{\eta^{19}(4z)\eta^{17}(6z)}{\eta^{11}(2z)\eta(12z)}$$

$$f_{16}:=\sum_{n=0}^{\infty} f_{16}(n)=\frac{\eta^{18}(4z)\eta^{18}(6z)}{\eta^{6}(2z)\eta^{6}(12z)}$$

$$f_{17}:=\sum_{n=0}^{\infty} f_{17}(n)q^n=\frac{\eta^{16}(4z)\eta^{8}(6z)\eta^{8}(12z)}{\eta^{8}(2z)}$$

$$f_{18}:=\sum_{n=0}^{\infty} f_{18}(n)q^n=\frac{\eta^{12}(2z)\eta^{12}(4z)\eta^{12}(12z)}{\eta^{12}(6z)}$$

$$f_{19}:=\sum_{n=0}^{\infty} f_{19}(n)q^n=\frac{\eta^{20}(4z)\eta^{16}(6z)\eta^{4}(12z)}{\eta^{16}(2z)}.$$

Now we can state our main Theorem:

Theorem 1. Let $b_1, b_2, \cdots, b_5$ be non-negative integers satisfying

$$b_1+b_2+\cdots+b_5\leq 24. \tag{5}$$

Define the integers $a_1, a_2, a_3, a_4, a_6, a_{12}$ by

$$a_1:=-b_1+2b_2-2b_3-4b_4-b_5+24, \tag{6}$$

$$a_2:=3b_1+b_2+3b_3+10b_4+b_5-60, \tag{7}$$

$$a_3:=3b_1+2b_2+6b_3+4b_4+3b_5-72, \tag{8}$$

$$a_4:=-2b_1-b_2-b_3-4b_4+2b_5+24, \tag{9}$$

$$a_6:=-9b_1-7b_2-9b_3-10b_4-7b_5+180, \tag{10}$$

$$a_{12}:=6b_1+3b_2+3b_3+4b_4+2b_5-72. \tag{11}$$

The functions $f_1, \ldots, f_{19}$ which are defined before, are functions of q by (3) and contained in $M_{12}(\Gamma_0(12))$. In fact, except f_9, f_{18}, and f_{19}, they are also cusp forms. Now define rational numbers

$k_0, k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8, k_9, k_{10}, k_{11}, k_{12},$
$k_{13}, k_{14}, k_{15}, k_{16}, k_{17}, k_{18}, k_{19}, k_{20}, k_{21}, k_{22}, k_{23}, k_{24}$

by

$$\frac{1}{2^{b_1+b_5}}x^{b_1}(1-x)^{b_2}(1+x)^{b_3}(1+2x)^{b_4}(2+x)^{b_5} \tag{12}$$

$$\begin{aligned}&=k_0+k_1x+k_2x^2+k_3x^3+k_4x^4+k_5x^5+k_6x^6+k_7x^7+k_8x^8+k_9x^9\\&+k_{10}x^{10}+k_{11}x^{11}+k_{12}x^{12}+k_{13}x^{13}+k_{14}x^{14}+k_{15}x^{15}+k_{16}x^{16}+k_{17}x^{17}\\&+k_{18}x^{18}+k_{19}x^{19}+k_{20}x^{20}+k_{21}x^{21}+k_{22}x^{22}+k_{23}x^{23}+k_{24}x^{24}.\end{aligned}$$

Define the rational numbers

$$c_1, c_2, c_3, c_4, c_6, c_{12}, r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, r_9, r_{10}, r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16}, r_{17}, r_{18}$$

and r_{19} as in Appendix. Then the Fourier coefficients of the following eta quotients

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z) = \delta(b_1) + \sum_{n=1}^{\infty} c\,(n)q^n,$$

can be given by the formula

$$c(n) = c_1\sigma_{11}(n) + c_2\sigma_{11}\left(\frac{n}{2}\right) + c_3\sigma_{11}\left(\frac{n}{3}\right) + c_4\sigma_{11}\left(\frac{n}{4}\right) + c_6\sigma_{11}\left(\frac{n}{6}\right) + c_{12}\sigma_{11}\left(\frac{n}{12}\right)$$
$$+r_1 f_1(n) + \ldots + r_{19} f_{19}(n), n \in \mathbb{N}.$$

In particular, for $n \in \mathbb{N}$, we have

$$c(2n) = c_1\sigma_{11}(2n) + c_2\sigma_{11}(n) + c_3\sigma_{11}\left(\frac{2n}{3}\right) + c_4\sigma_{11}\left(\frac{n}{2}\right) + c_6\sigma_{11}\left(\frac{n}{3}\right) + c_{12}\sigma_{11}\left(\frac{n}{6}\right)$$
$$+r_{11} f_{11}(2n) + \ldots + r_{19} f_{19}(2n),$$

$$c(2n-1) = c_1\sigma_{11}(2n-1)$$
$$+c_3\sigma_{11}\left(\frac{2n-1}{3}\right) + r_1 f_1(2n-1) + r_2 f_2(2n-1) + \ldots + r_{10} f_{10}(2n-1),$$

and

$$f_{11}(2n-1) = f_{12}(2n-1) = \ldots f_{19}(2n-1) = 0,$$

$$f_1(2n) = f_2(2n) = \ldots = f_{10}(2n) = 0.$$

Proof. It follows from

(6-11) that

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 12a_{12} = 24b_1 \tag{13}$$

$$a_1 + a_2 + a_3 + a_4 + a_6 + a_{12} = 24, \tag{14}$$

$$-\frac{a_1}{6} - \frac{a_2}{3} - \frac{a_3}{6} - \frac{2a_4}{3} - \frac{a_6}{3} - \frac{2a_{12}}{3} = -b_1 - b_5. \tag{15}$$

Now we will use p-k parametrization of Alaca, Alaca and Williams, see [15]:

$$p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{16}$$

where the theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{-\infty}^{\infty} q^{n^2}.$$

Setting x=p in (12), and multiplying both sides by k^12,we obtain

$$\frac{k^{12}}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}$$

$$= (k_0 + k_1p + k_2p^2 + k_3p^3 + k_4p^4 + k_5p^5 + k_6p^6 + k_7p^7 + k_8p^8 + k_9p^9 + k_{10}p^{10} + k_{11}p^{11} + k_{12}p^{12} + k_{13}p^{13} + k_{14}p^{14} + k_{15}p^{15} + k_{16}p^{16} + k_{17}p^{17} + k_{18}p^{18} + k_{19}p^{19} + k_{20}p^{20} + k_{21}p^{21} + k_{22}p^{22} + k_{23}p^{23} + k_{24}p^{24})k^{12}.$$

Alaca, Alaca and Williams [16] have established the following representations in terms of p and k:

$$\eta(q) = 2^{-1/6}p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8}k^{1/2}, \tag{17}$$

$$\eta(q^2) = 2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2}, \tag{18}$$

$$\eta(q^3) = 2^{-1/6}p^{1/8}(1-p)^{1/6}(1+p)^{1/2}(1+2p)^{1/24}(2+p)^{1/24}k^{1/2}, \tag{19}$$

$$\eta(q^4) = 2^{-2/3}p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2}, \tag{20}$$

$$\eta(q^6) = 2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2}, \tag{21}$$

$$\eta(q^{12}) = 2^{-2/3}p^{1/2}(1-p)^{1/24}(1+p)^{1/8}(1+2p)^{1/24}(2+p)^{1/6}k^{1/2}. \tag{22}$$

Moreover, they have obtained similar formulas for E_4 and E_6. Now, the formulas in the Appendix for E_{12} can be obtained from the well-known formula

$$691E_{12} = 441E_4^3 + 250E_6^2.$$

Obviously, $f_1, \ldots f_{19}$ are functions of q , see (3),(16), contained in $M_{12}(\Gamma_0(12))$. In fact, except f_9, f_{18}, and f_{19}, they are also cusp forms and $ord_{1/3}f_9 = ord_{1/6}f_9 = ord_{1/3}f_{18} = ord_{1/6}f_{18} = ord_{1/1}f_{19} = ord_{1/2}f_{19} = 0$, by [17]. Now using formulas 17,18,19,20,21 and 22, we get

$$\eta^{a_1}(z)\eta^{a_2}(2z)\eta^{a_3}(3z)\eta^{a_4}(4z)\eta^{a_6}(6z)\eta^{a_{12}}(12z)$$
$$= q^{b_1}\prod_{n=1}^{\infty}(1-q^n)^{a_1}(1-q^{2n})^{a_2}(1-q^{3n})^{a_3}(1-q^{4n})^{a_4}(1-q^{6n})^{a_6}(1-q^{12n})^{a_{12}}$$
$$= 2^{-\frac{a_1}{6}-\frac{a_2}{3}-\frac{a_3}{6}-\frac{2a_4}{3}-\frac{a_6}{3}-\frac{2a_{12}}{3}}p^{\frac{a_1}{24}+\frac{a_2}{12}+\frac{a_3}{8}+\frac{a_4}{6}+\frac{a_6}{4}+\frac{a_{12}}{2}}(1-p)^{\frac{a_1}{2}+\frac{a_2}{4}+\frac{a_3}{6}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}$$
$$(1+p)^{\frac{a_1}{6}+\frac{a_2}{12}+\frac{a_3}{2}+\frac{a_4}{24}+\frac{a_6}{4}+\frac{a_{12}}{8}}(1+2p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{8}+\frac{a_6}{12}+\frac{a_{12}}{24}}$$
$$(2+p)^{\frac{a_1}{8}+\frac{a_2}{4}+\frac{a_3}{24}+\frac{a_4}{2}+\frac{a_6}{12}+\frac{a_{12}}{6}}k^{\frac{a_1+a_2+a_3+a_4+a_6+a_{12}}{2}}$$
$$= \frac{k^{12}}{2^{b_1+b_5}}p^{b_1}(1-p)^{b_2}(1+p)^{b_3}(1+2p)^{b_4}(2+p)^{b_5}$$

$$= k^{12}(k_0 + k_1p + k_2p^2 + k_3p^3 + k_4p^4 + k_5p^5 + k_6p^6 + k_7p^7 + k_8p^8 + k_9p^9 + k_{10}p^{10} + k_{11}p^{11} + k_{12} p^{12} + k_{13}p^{13} + k_{14}p^{14} + k_{15}p^{15} + k_{16}p^{16} + k_{17}p^{17} + k_{18} p^{18} + k_{19}p^{19} + k_{20}p^{20} + k_{21}p^{21} + k_{22}p^{22} + k_{23}p^{23} + k_{24}p^{24})$$

$$= \frac{691c_1}{65520}\left(1 + \frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n\right)$$
$$+ \frac{691c_2}{65520}\left(1 + \frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n}\right)$$

$$+\frac{691c_3}{65520}(1+\frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{3n})$$
$$+\frac{691c_4}{65520}(1+\frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n})$$
$$+\frac{691c_6}{65520}(1+\frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{6n})$$
$$+\frac{691c_{12}}{65520}(1+\frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{12n})$$
$$+r_1q^7\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{20}(1-q^{6n})^{10}(1-q^{12n})^4}{(1-q^{2n})^{10}}$$
$$+r_2q^7\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{15}(1-q^{6n})^{15}(1-q^{12n})^3}{(1-q^{2n})^9}$$
$$+r_3q^7\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{10}(1-q^{6n})^{20}(1-q^{12n})^2}{(1-q^{2n})^8}$$
$$+r_4q^5\prod_{n=1}^{\infty}(1-q^{2n})^5(1-q^{4n})^5(1-q^{6n})^{13}(1-q^{12n})$$
$$+r_5q^7\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{16}(1-q^{6n})^2(1-q^{12n})^8}{(1-q^{2n})^2}$$
$$+r_6q^7\prod_{n=1}^{\infty}\frac{(1-q^{2n})^9(1-q^{6n})^9(1-q^{12n})^9}{(1-q^{4n})^3}$$
$$+r_7q^5\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{11}(1-q^{4n})^{11}(1-q^{12n})^7}{(1-q^{6n})^5}$$
$$+r_8q^{11}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{12}(1-q^{12n})^{18}}{(1-q^{4n})^6}$$
$$+r_9q^7\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{14}(1-q^{4n})^8(1-q^{12n})^{16}}{(1-q^{6n})^{14}}$$
$$+r_{10}q\prod_{n=1}^{\infty}\frac{(1-q^{2n})^6(1-q^{4n})^{12}(1-q^{6n})^{18}}{(1-q^{12n})^{12}}$$
$$+r_{11}q^4\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{20}(1-q^{6n})^4(1-q^{12n})^4}{(1-q^{4n})^4}$$
$$+r_{12}q^4\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{19}(1-q^{4n})(1-q^{12n})^5}{(1-q^{6n})}$$
$$+r_{13}q^6\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{20}(1-q^{4n})^4(1-q^{12n})^4}{(1-q^{2n})^4}$$
$$+r_{14}q^{10}\prod_{n=1}^{\infty}\frac{(1-q^{6n})^{14}(1-q^{12n})^{14}}{(1-q^{2n})^2(1-q^{4n})^2}$$
$$+r_{15}q^6\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{19}(1-q^{6n})^{17}}{(1-q^{2n})^{11}(1-q^{12n})}$$

$$+r_{16}q^4\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{18}(1-q^{6n})^{18}}{(1-q^{2n})^6(1-q^{12n})^6}$$
$$+r_{17}q^8\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{16}(1-q^{6n})^{8}(1-q^{12n})^8}{(1-q^{2n})^8}$$
$$+r_{18}q^6\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{11}(1-q^{4n})^{12}(1-q^{12n})^{12}}{(1-q^{6n})^{12}}$$
$$+r_{19}q^8\prod_{n=1}^{\infty}\frac{(1-q^{4n})^{20}(1-q^{6n})^{16}(1-q^{12n})^4}{(1-q^{2n})^{16}}$$

$$= \delta(b_1)\sum_{n=1}^{\infty}\quad (c_1(\sigma_{11}(n)+c_2(\sigma_{11}\left(\frac{n}{2}\right)$$
$$+c_3\sigma_5\left(\frac{n}{3}\right)+c_4(\sigma_5\left(\frac{n}{4}\right)$$
$$+c_6\sigma_5\left(\frac{n}{6}\right)+c_{12}\sigma_5\left(\frac{n}{12}\right))$$
$$+r_1f_1(n)+\ldots.+r_{19}f_{19}(n),$$

where the constant term of the Fourier series is

$$\delta(b_1)=\begin{cases}0 \text{ if } \mathrm{b}_1\neq 0\\ 1 \text{ if } \mathrm{b}_1=0.\end{cases}$$

So

$$c(n)=c_1\sigma_{11}(n)+c_2\sigma_{11}\left(\frac{n}{2}\right)+c_3\sigma_{11}\left(\frac{n}{3}\right)+c_4\sigma_{11}\left(\frac{n}{4}\right)+c_6\sigma_{11}\left(\frac{n}{6}\right)$$
$$+c_{12}\sigma_{11}\left(\frac{n}{12}\right)+r_1f_1(n)+\ldots.+r_{19}f_{19}(n).$$

In particular,

$$c(2n)=c_1\sigma_{11}(2n)+c_2\sigma_{11}(n)+c_3\left(2049\sigma_{11}\left(\frac{n}{3}\right)-2048\sigma_{11}\left(\frac{n}{6}\right)\right)$$
$$+c_4\sigma_{11}\left(\frac{n}{2}\right)+c_6\sigma_{11}\left(\frac{n}{3}\right)+c_{12}\sigma_{11}\left(\frac{n}{6}\right)+r_{11}f_{11}(2n)+\ldots.+r_{19}f_{19}(2n).$$
$$=c_1\sigma_{11}(2n)+c_2\sigma_{11}(n)+c_4\sigma_{11}\left(\frac{n}{2}\right)+(2049c_3+c_6)\sigma_{11}\left(\frac{n}{3}\right)$$
$$+(c_{12}-2048c_3)\sigma_{11}\left(\frac{n}{6}\right),$$

if $r_{11}=r_{12}=\cdots=r_{19}=0$.

$$c(2n-1)=c_1\sigma_{11}(2n-1)+c_3\sigma_{11}\left(\frac{2n-1}{3}\right)$$
$$+r_1f_1(2n-1)+r_2f_2(2n-1)+\ldots+r_{10}f_{10}(2n-1),$$

since for n=1,2,⋯,

$$f_{11}(2n-1)=f_{12}(2n-1)=\ldots.f_{19}(2n-1)=0,$$

$$f_1(2n)=f_2(2n)=\ldots=f_{10}(2n)=0,$$

and it is easy to see that

$$\sigma_{11}\left(\frac{2n}{3}\right) = (2^{11}+1)\sigma_{11}\left(\frac{n}{3}\right) - 2^{11}\sigma_{11}\left(\frac{n}{6}\right)$$
$$= 2049\sigma_{11}\left(\frac{n}{3}\right) - 2048\sigma_{11}\left(\frac{n}{6}\right).$$

Remark 1. These formulas are valid for 112116 nontrivial eta quotients. Among them, we have found 198 eta quotients, (see Table 1, Supplementary) such that

$$c(2n) = c_1\sigma_{11}(2n) + c_2\sigma_{11}(n) + c_4\sigma_{11}\left(\frac{n}{2}\right) + (2049c_3 + c_6)\sigma_{11}\left(\frac{n}{3}\right)$$
$$+(c_{12} - 2048c_3)\sigma_{11}\left(\frac{n}{6}\right)$$

since $f_1(2n) = f_2(2n) = \ldots = f_{10}(2n) = 0$ and $r_{11} = r_{12} = \cdots = r_{19} = 0.$ We have also found 334 eta quotients, (see Table 2, Supplementary) such that

$$c(2n-1) = c_1\sigma_{11}(2n-1) + c_3\sigma_{11}\left(\frac{2n-1}{3}\right) = 0$$

since $f_{11}(2n-1) = f_{12}(2n-1) = \ldots . f_{19}(2n-1) = 0$ and $r_1 = r_2 = \cdots = r_{10} = 0.$

Remark 2. If f is an eta quotient, then $f(-q)$ is also an eta quotient, and the coefficients of $\frac{1}{2}(f(q) + f(-q))$ are exactly the even coefficients of f. In particular, it means that we have obtained all coefficients of the the sum of 198 eta quotients. Moreover, $\frac{1}{2}(f(q) - f(-q))$ are exactly the odd coefficients of f. In particular, it means that $\frac{1}{2}(f(q) - f(-q)) = 0$ for 334 eta quotients.

Remark 3. $S_{12}(\Gamma_0(12))$ is 19 dimensional, see [18] (Chapter 3, pg.87 and Chapter 5, pg.197), and generated by

$$\Delta, \Delta(2z), \Delta(3z), \Delta(4z), \Delta(6z), \Delta(12z), \Delta_{3,12}, \Delta_{3,12}(2z),$$
$$\Delta_{3,12}(4z), \Delta_{4,12}, \Delta_{4,12}(3z), \Delta_{6,12,1}, \Delta_{6,12,1}(2z), \Delta_{6,12,2},$$
$$\Delta_{6,12,2}(2z), \Delta_{6,12,3}, \Delta_{6,12,3}(2z), \Delta_{12,12,1}, \Delta_{12,12,2}$$

where Δ is the unique cuspidal form in $S_{12}(\Gamma_0(1))$, $\Delta_{3,12}$ is the unique newform in $S_{12}(\Gamma_0(3))$, $\Delta_{4,12}$ is the unique newform in

$S_{12}(\Gamma_0(4))$, $\Delta_{6,12,1}, \Delta_{6,12,2}, \Delta_{6,12,3}$ are all newforms in $S_{12}(\Gamma_0(6))$ and $\Delta_{12,12,1}, \Delta_{12,12,2}$ are all newforms in $S_{12}(\Gamma_0(12))$. By simple calculation, we see that

$$f_1 = -\frac{79}{18800640}\Delta(z) - \frac{79}{783360}\Delta(2z) + \frac{7353}{696320}\Delta(3z) - \frac{79}{9180}\Delta(4z)$$
$$+\frac{22059}{87040}\Delta(6z) + \frac{7353}{340}\Delta(12z) + \frac{401}{27260928}\Delta_{3,12}(z) - \frac{5213}{4543488}\Delta_{3,12}(2z)$$
$$+\frac{401}{13311}\Delta_{3,12}(4z) - \frac{13}{1400832}\Delta_{4,12}(z) - \frac{2079}{155648}\Delta_{4,12}(3z)$$
$$-\frac{13}{1105920}\Delta_{6,12,1}(z) + \frac{13}{34560}\Delta_{6,12,1}(2z) - \frac{1}{165888}\Delta_{6,12,2}(z)$$
$$-\frac{1}{5184}\Delta_{6,12,2}(2z) + \frac{35}{4810752}\Delta_{6,12,3}(z) + \frac{35}{150336}\Delta_{6,12,3}(2z)$$
$$+\frac{5}{221184}\Delta_{12,12,1}(z) - \frac{7}{525312}\Delta_{12,12,2}(z)$$

$$f_2=-\frac{13}{3133440}\Delta(z)-\frac{13}{130560}\Delta(2z)+\frac{3333}{348160}\Delta(3z)-\frac{13}{1530}\Delta(4z)$$
$$+\frac{9999}{43520}\Delta(6z)+\frac{3333}{170}\Delta(12z)+\frac{173}{13630464}\Delta_{3,12}(z)$$
$$-\frac{2249}{2271744}\Delta_{3,12}(2z)+\frac{346}{13311}\Delta_{3,12}(4z)-\frac{1}{126976}\Delta_{4,12}(z)$$
$$-\frac{1503}{126976}\Delta_{4,12}(3z)-\frac{11}{1105920}\Delta_{6,12,1}(z)+\frac{11}{34560}\Delta_{6,12,1}(2z)$$
$$-\frac{11}{1990656}\Delta_{6,12,2}(z)-\frac{11}{62208}\Delta_{6,12,2}(2z)+\frac{25}{3608064}\Delta_{6,12,3}(z)$$
$$+\frac{25}{112752}\Delta_{6,12,3}(2z)+\frac{205}{10285056}\Delta_{12,12,1}(z)-\frac{1}{82944}\Delta_{12,12,2}(z)$$

$$f_3=-\frac{7}{1762560}\Delta(z)-\frac{7}{73440}\Delta(2z)+\frac{559}{65280}\Delta(3z)$$
$$-\frac{56}{6885}\Delta(4z)+\frac{559}{2720}\Delta(6z)+\frac{4472}{255}\Delta(12z)$$
$$+\frac{7}{638928}\Delta_{3,12}(z)-\frac{91}{106488}\Delta_{3,12}(2z)$$
$$+\frac{896}{39933}\Delta_{3,12}(4z)-\frac{1}{150784}\Delta_{4,12}(z)-\frac{1565}{150784}\Delta_{4,12}(3z)$$
$$-\frac{7}{829440}\Delta_{6,12,1}(z)+\frac{7}{25920}\Delta_{6,12,1}(2z)-\frac{23}{4478976}\Delta_{6,12,2}(z)$$
$$-\frac{23}{139968}\Delta_{6,12,2}(2z)+\frac{107}{16236288}\Delta_{6,12,3}(z)$$
$$+\frac{107}{507384}\Delta_{6,12,3}(2z)+\frac{17}{964224}\Delta_{12,12,1}(z)-\frac{13}{1181952}\Delta_{12,12,2}(z)$$

$$f_4=\frac{1}{32640}\Delta(z)+\frac{1}{1360}\Delta(2z)-\frac{63}{10880}\Delta(3z)+\frac{16}{255}\Delta(4z)$$
$$-\frac{189}{1360}\Delta(6z)-\frac{1008}{85}\Delta(12z)-\frac{1}{212976}\Delta_{3,12}(z)$$
$$+\frac{13}{35496}\Delta_{3,12}(2z)-\frac{128}{13311}\Delta_{3,12}(4z)-\frac{3}{75392}\Delta_{4,12}(z)$$
$$-\frac{1161}{75392}\Delta_{4,12}(3z)-\frac{1}{138240}\Delta_{6,12,1}(z)+\frac{1}{4320}\Delta_{6,12,1}(2z)$$
$$-\frac{1}{82944}\Delta_{6,12,2}(z)-\frac{1}{2592}\Delta_{6,12,2}(2z)-\frac{1}{150336}\Delta_{6,12,3}(z)$$
$$-\frac{1}{4698}\Delta_{6,12,3}(2z)+\frac{1}{107136}\Delta_{12,12,1}(z)+\frac{1}{32832}\Delta_{12,12,2}(z)$$

$$f_5=-\frac{19}{3760128}\Delta(z)-\frac{19}{156672}\Delta(2z)+\frac{405}{139264}\Delta(3z)$$
$$-\frac{19}{1836}\Delta(4z)+\frac{1215}{17408}\Delta(6z)+\frac{405}{68}\Delta(12z)$$
$$-\frac{47}{18173952}\Delta_{3,12}(z)+\frac{611}{3028992}\Delta_{3,12}(2z)$$
$$-\frac{47}{8874}\Delta_{3,12}(4z)+\frac{157}{43425792}\Delta_{4,12}(z)$$
$$-\frac{2673}{4825088}\Delta_{4,12}(3z)+\frac{1}{147456}\Delta_{6,12,1}(z)$$
$$-\frac{1}{4608}\Delta_{6,12,1}(2z)-\frac{1}{442368}\Delta_{6,12,2}(z)-\frac{1}{13824}\Delta_{6,12,2}(2z)$$
$$+\frac{5}{1603584}\Delta_{6,12,3}(z)+\frac{5}{50112}\Delta_{6,12,3}(2z)+$$
$$-\frac{5}{2285568}\Delta_{12,12,1}(z)-\frac{1}{700416}\Delta_{12,12,2}(z)$$

$$f_6 = \frac{1}{32640}\Delta(z) + \frac{1}{1360}\Delta(2z) - \frac{63}{10880}\Delta(3z) + \frac{16}{255}\Delta(4z) - \frac{189}{1360}\Delta(6z) - \frac{1008}{85}\Delta(12z) - \frac{1}{212976}\Delta_{3,12}(z) + \frac{13}{35496}\Delta_{3,12}(2z) - \frac{128}{13311}\Delta_{3,12}(4z) + \frac{1}{75392}\Delta_{4,12}(z) + \frac{387}{75392}\Delta_{4,12}(3z) - \frac{1}{138240}\Delta_{6,12,1}(z) + \frac{1}{4320}\Delta_{6,12,1}(2z) - \frac{1}{82944}\Delta_{6,12,2}(z) - \frac{1}{2592}\Delta_{6,12,2}(2z) - \frac{1}{150336}\Delta_{6,12,3}(z) - \frac{1}{4698}\Delta_{6,12,3}(2z) - \frac{1}{321408}\Delta_{12,12,1}(z) - \frac{1}{98496}\Delta_{12,12,2}(z)$$

$$f_7 = \frac{211}{626688}\Delta(z) + \frac{211}{26112}\Delta(2z) - \frac{2187}{69632}\Delta(3z) + \frac{211}{306}\Delta(4z) - \frac{6561}{8704}\Delta(6z) - \frac{2187}{34}\Delta(12z) + \frac{213}{504832}\Delta_{3,12}(z) - \frac{8307}{252416}\Delta_{3,12}(2z) + \frac{426}{493}\Delta_{3,12}(4z) - \frac{3517}{7237632}\Delta_{4,12}(z) - \frac{531441}{2412544}\Delta_{4,12}(3z) + \frac{3}{8192}\Delta_{6,12,1}(z) - \frac{3}{256}\Delta_{6,12,1}(2z) + \frac{7}{73728}\Delta_{6,12,2}(z) + \frac{7}{2304}\Delta_{6,12,2}(2z) + \frac{11}{133632}\Delta_{6,12,3}(z) + \frac{11}{4176}\Delta_{6,12,3}(2z) + - \frac{71}{126976}\Delta_{12,12,1}(z) - \frac{5}{19456}\Delta_{12,12,2}(z)$$

$$f_8 = \frac{1}{352512}\Delta(z) + \frac{1}{14688}\Delta(2z) - \frac{259}{39168}\Delta(3z) + \frac{8}{1377}\Delta(4z) - \frac{259}{1632}\Delta(6z) - \frac{2072}{153}\Delta(12z) - \frac{5}{638928}\Delta_{3,12}(z) + \frac{65}{106488}\Delta_{3,12}(2z) - \frac{640}{39933}\Delta_{3,12}(4z) - \frac{37}{12213504}\Delta_{4,12}(z) - \frac{17731}{4071168}\Delta_{4,12}(3z) + \frac{1}{165888}\Delta_{6,12,1}(z) - \frac{1}{5184}\Delta_{6,12,1}(2z) + \frac{7}{1492992}\Delta_{6,12,2}(z) + \frac{7}{46656}\Delta_{6,12,2}(2z) - \frac{31}{5412096}\Delta_{6,12,3}(z) - \frac{31}{169128}\Delta_{6,12,3}(2z) + \frac{7}{964224}\Delta_{12,12,1}(z) - \frac{5}{1181952}\Delta_{12,12,2}(z)$$

$$f_9 = \frac{1543673}{43304140080}\Delta(z) + \frac{1543673}{180433920}\Delta(2z) - \frac{38051613}{481157120}\Delta(3z) + \frac{1543673}{2114460}\Delta(4z) - \frac{1141548339}{60144640}\Delta(6z) - \frac{38051613}{234940}\Delta(12z) + \frac{46641}{147410944}\Delta_{3,12}(z) - \frac{1818999}{73705472}\Delta_{3,12}(2z) + \frac{46641}{71978}\Delta_{3,12}(4z) - \frac{63891}{14475264}\Delta_{4,12}(z) - \frac{767637}{4825088}\Delta_{4,12}(3z) + \frac{141}{573440}\Delta_{6,12,1}(z) - \frac{141}{17920}\Delta_{6,12,1}(2z) + \frac{331}{1916928}\Delta_{6,12,2}(z) + \frac{331}{59904}\Delta_{6,12,2}(2z) + \frac{563}{2672640}\Delta_{6,12,3}(z) + \frac{563}{83520}\Delta_{6,12,3}(2z) + - \frac{137}{253952}\Delta_{12,12,1}(z)$$

$$-\frac{25}{77824}\Delta_{12,12,2}(z)+\frac{1}{4457361592320}E_{12}(z)$$
$$-\frac{683}{1485787197440}E_{12}(2z)-\frac{177147}{1485787197440}E_{12}(3z)$$
$$+\frac{1}{2176446090}E_{12}(4z)+\frac{362974203}{1485787197440}E_{12}(6z)$$
$$-\frac{177147}{725482030}E_{12}(12z)$$

$$f_{10}=\frac{2201}{12240}\Delta(z)+\frac{2201}{510}\Delta(2z)-\frac{50301}{1360}\Delta(3z)+\frac{281728}{765}\Delta(4z)$$
$$-\frac{150903}{170}\Delta(6z)-\frac{6438528}{85}\Delta(12z)+\frac{155}{986}\Delta_{3,12}(z)$$
$$-\frac{6045}{493}\Delta_{3,12}(2z)+\frac{158720}{493}\Delta_{3,12}(4z)+\frac{6475}{56544}\Delta_{4,12}(z)$$
$$+\frac{8007003}{18848}\Delta_{4,12}(3z)+\frac{79}{640}\Delta_{6,12,1}(z)-\frac{79}{20}\Delta_{6,12,1}(2z)$$
$$+\frac{109}{1152}\Delta_{6,12,2}(z)+\frac{109}{36}\Delta_{6,12,2}(2z)+\frac{233}{2088}\Delta_{6,12,3}(z)$$
$$+\frac{932}{261}\Delta_{6,12,3}(2z)+\frac{413}{2976}\Delta_{12,12,1}(z)+\frac{73}{912}\Delta_{12,12,2}(z)$$

$$f_{11}=-\frac{26}{765}\Delta(2z)-\frac{896}{765}\Delta(4z)-\frac{729}{85}\Delta(6z)+\frac{279936}{85}\Delta(12z)$$
$$+\frac{113}{2958}\Delta_{3,12}(2z)-\frac{1024}{1479}\Delta_{3,12}(4z)-\frac{1}{60}\Delta_{6,12,1}(2z)$$
$$+\frac{1}{36}\Delta_{6,12,2}(2z)-\frac{4}{261}\Delta_{6,12,3}(2z)$$

$$f_{12}=-\frac{1027}{12240}\Delta(2z)-\frac{112}{765}\Delta(4z)+\frac{2187}{1360}\Delta(6z)+\frac{34992}{85}\Delta(12z)$$
$$+\frac{39}{1972}\Delta_{3,12}(2z)-\frac{768}{493}\Delta_{3,12}(4z)+\frac{3}{160}\Delta_{6,12,1}(2z)$$
$$+\frac{7}{288}\Delta_{6,12,2}(2z)+\frac{11}{522}\Delta_{6,12,3}(2z)$$

$$f_{13}=-\frac{7}{24480}\Delta(2z)-\frac{26}{765}\Delta(4z)+\frac{2187}{2720}\Delta(6z)-\frac{729}{85}\Delta(12z)$$
$$-\frac{1}{5916}\Delta_{3,12}(2z)+\frac{113}{2958}\Delta_{3,12}(4z)+\frac{1}{3840}\Delta_{6,12,1}(2z)$$
$$+\frac{1}{2304}\Delta_{6,12,2}(2z)-\frac{1}{4176}\Delta_{6,12,3}(2z)+$$

$$f_{14}=\frac{1}{36}\Delta(6z)+\frac{16}{9}\Delta(12z)-\frac{1}{17496}\Delta_{6,12,2}(2z)$$
$$+\frac{1}{17496}\Delta_{6,12,3}(2z)$$

$$f_{15}=-\frac{7}{24480}\Delta(2z)-\frac{13}{170}\Delta(4z)+\frac{2187}{2720}\Delta(6z)+\frac{29997}{170}\Delta(12z)$$
$$-\frac{5}{1836}\Delta_{3,12}(2z)+\frac{98}{459}\Delta_{3,12}(4z)+\frac{89}{34560}\Delta_{6,12,1}(2z)$$
$$-\frac{13}{6912}\Delta_{6,12,2}(2z)+\frac{1}{432}\Delta_{6,12,3}(2z)$$

$$f_{16} = -\frac{7}{4080}\Delta(2z) + \frac{36}{85}\Delta(4z) + \frac{6561}{1360}\Delta(6z) - \frac{6804}{85}\Delta(12z) + \frac{11}{5916}\Delta_{3,12}(2z) + \frac{176}{1479}\Delta_{3,12}(4z) + \frac{1}{240}\Delta_{6,12,1}(2z) - \frac{1}{232}\Delta_{6,12,3}(2z)$$

$$f_{17} = -\frac{1}{153}\Delta(4z) + \frac{378}{17}\Delta(12z) - \frac{1}{3132}\Delta_{3,12}(2z) + \frac{308}{13311}\Delta_{3,12}(4z) + \frac{1}{3456}\Delta_{6,12,1}(2z) - \frac{1}{3456}\Delta_{6,12,2}(2z) + \frac{1}{3132}\Delta_{6,12,3}(2z) +$$

$$f_{18} = -\frac{717}{110560}\Delta(2z) - \frac{189277}{58735}\Delta(4z) + \frac{216513}{110560}\Delta(6z) + \frac{37712628}{58735}\Delta(12z) + \frac{297}{8468}\Delta_{3,12}(2z) - \frac{194697}{71978}\Delta_{3,12}(4z) + \frac{297}{8960}\Delta_{6,12,1}(2z) - \frac{95}{3328}\Delta_{6,12,2}(2z) - \frac{77}{2320}\Delta_{6,12,3}(2z) + -\frac{1}{2176446090}E_{12}(2z) + \frac{1}{2176446090}E_{12}(4z) + \frac{177147}{725482030}E_{12}(6z) - \frac{177147}{725482030}E_{12}(12z)$$

$$f_{19} = \frac{31}{1585845}\Delta(2z) + \frac{13067}{3171690}\Delta(4z) - \frac{3277}{58735}\Delta(6z) - \frac{1142579}{117470}\Delta(12z) + \frac{49}{323901}\Delta_{3,12}(2z) - \frac{7909}{647802}\Delta_{3,12}(4z) - \frac{1}{6720}\Delta_{6,12,1}(2z) + \frac{61}{606528}\Delta_{6,12,2}(2z) - \frac{13}{105705}\Delta_{6,12,3}(2z) + \frac{3}{725482030}E_{12}(2z) - \frac{3}{725482030}E_{12}(4z) - \frac{3}{725482030}E_{12}(6z) + \frac{3}{725482030}E_{12}(12z)$$

2 Conclusıon

We have found 198 eta quotients of weight 12, (see Table 1, Supplementary) such that

$c(2n) = c_1\sigma_{11}(2n) + c_2\sigma_{11}(n) + c_4\sigma_{11}\left(\frac{n}{2}\right) + (2049c_3 + c_6)\sigma_{11}\left(\frac{n}{3}\right) + (c_{12} - 2048c_3)\sigma_{11}\left(\frac{n}{6}\right)$ and 334 eta quotients of weight 12, (see Table 2, Supplementary) such that

$$c(2n-1) = c_1\sigma_{11}(2n-1) + c_3\sigma_{11}\left(\frac{2n-1}{3}\right) = 0.$$

Competıng Interests

Author has declared that no competing interests exist.

References

[1] Williams KS. Fourier series of a class of eta quotients. Int. J. Number Theory. 2012;8:993-1004.

[2] Olivia XM Yao, Ernest XW Xia, Jin J. Explicit formulas for the Fourier coefficients of a class of eta quotients. International Journal of Number Theory. 2013;9(2):487-503.

[3] Köhler G. Eta products and theta series ıdentities. Springer−Verlag, Berlin; 2011.

[4] Gordon B. Some identities in combinatorial analysis. Quart. J. Math. Oxford Ser. 1961;12:285-290.

[5] Kac VG. Infinite−dimensional algebras, Dedekind's η −function, classical Möbius function and the very strange formula. Adv. Math. 1978;30:85-136.

[6] Macdonald IG. Affine root systems and Dedekind's η −function. Invent. Math. 1972;15:91-143.

[7] Zucker IJ. A systematic way of converting infinite series into infinite products. J. Phys. 1987;A20: L13-L17.

[8] Zucker IJ. Further relations amongst infinite series and products: II. The evaluation of three−dimensional lattice sums. J. Phys. 1990;A23:117-132.

[9] Kendirli B. Evaluation of Some Convolution Sums by Quasimodular Forms. European Journal of Pure and Applied Mathematics. 2015;8(1):81-110. ISSN: 13075543.

[10] Kendirli B. Evaluation of some convolution sums and representation numbers of quadratic forms of discriminant 135. British Journal of Mathematics and Computer Science. 2015;6/6:494-531.

[11] Kendirli B. Evaluation of some convolution sums and the representation numbers, Ars Combinatorica Volume CXVI. 2015;65-91.

[12] Kendirli B. Cusp forms in $S_4(\Gamma_0(79))$ and the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -79. Bulletin of the Korean Mathematical Society. 2012;49/3.

[13] Kendirli B. Cusp forms in $S_4(\Gamma_0(47))$ and the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -47, Hindawi. International Journal of Mathematics and Mathematical Sciences. 2012;303492:10.

[14] Kendirli B. The bases of $M_4(\Gamma_0(71)), M_6(\Gamma_0(71))$ and the number of representations of ıntegers, hindawi. Mathematical Problems in Engineering. 2013;695265:34.

[15] Alaca A, Alaca S, Williams KS. On the two−dimensional theta functions of borweins. Acta Arith. 2006;124:177-195.

[16] Alaca A, Alaca S, Williams KS. Evaluation of the convolution sums $\sum_{l+12m=n} \sigma(l)\sigma(m)$ and $\sum_{3l+4m=n} \sigma(l)\sigma(m)$, Adv. Theor. Appl. Math. 2006;1:27−48.

[17] Gordon B, Robins S. Lacunarity of dedekind η −products. Glasgow Math. J. 1995;37:1-14.

[18] Diamond F, Shurman J. A first course in modular forms. Springer Graduate Texts in Mathematics. 2015;228.

Appendix

$$c_1 := \frac{137191304}{19829797677}k_0 - \frac{53657770}{59489393031}k_1 - \frac{7593272}{59489393031}k_2 + \frac{3280504}{19829797677}k_3$$
$$- \frac{5592400}{59489393031}k_4 + \frac{2796208}{59489393031}k_5 - \frac{155344}{6609932559}k_6 + \frac{699056}{59489393031}k_7$$
$$- \frac{349520}{59489393031}k_8 + \frac{58256}{19829797677}k_9 - \frac{87376}{59489393031}k_{10} + \frac{43696}{59489393031}k_{11}$$
$$- \frac{7280}{19829797677}k_{12} + \frac{10928}{59489393031}k_{13} - \frac{5456}{59489393031}k_{14} + \frac{304}{6609932559}k_{15}$$
$$- \frac{1360}{59489393031}k_{16} + \frac{688}{59489393031}k_{17} - \frac{112}{19829797677}k_{18} + \frac{176}{59489393031}k_{19}$$
$$- \frac{80}{59489393031}k_{20} + \frac{16}{19829797677}k_{21} - \frac{16}{59489393031}k_{22} + \frac{16}{59489393031}k_{23}$$

$$c_2 := \frac{1968597297776}{6609932559}k_0 - \frac{875522436232}{19829797677}k_1 + \frac{84505395484}{19829797677}k_2 - \frac{2240584232}{6609932559}k_3$$
$$+ \frac{3819609200}{19829797677}k_4 - \frac{1909810064}{19829797677}k_5 + \frac{106099952}{2203310853}k_6 - \frac{477455248}{19829797677}k_7$$
$$+ \frac{238722160}{19829797677}k_8 - \frac{39788848}{6609932559}k_9 + \frac{59677808}{19829797677}k_{10} - \frac{29844368}{19829797677}k_{11}$$
$$+ \frac{4972240}{6609932559}k_{12} - \frac{7463824}{19829797677}k_{13} + \frac{3726448}{19829797677}k_{14} - \frac{207632}{2203310853}k_{15}$$
$$+ \frac{928880}{19829797677}k_{16} - \frac{469904}{19829797677}k_{17} + \frac{76496}{6609932559}k_{18} - \frac{120208}{19829797677}k_{19}$$
$$+ \frac{54640}{19829797677}k_{20} - \frac{10928}{6609932559}k_{21} + \frac{10928}{19829797677}k_{22} - \frac{10928}{19829797677}k_{23}$$

$$c_3 := -\frac{83476104536984}{19829797677}k_0 + \frac{44366472902746}{59489393031}k_1 - \frac{3889888078840}{59489393031}k_2 - \frac{422516024200}{19829797677}k_3$$
$$+ \frac{990716029600}{59489393031}k_4 - \frac{495359432032}{59489393031}k_5 + \frac{27519810976}{6609932559}k_6 - \frac{123840566624}{59489393031}k_7$$
$$+ \frac{61918866080}{59489393031}k_8 - \frac{10320283424}{19829797677}k_9 + \frac{15479007904}{59489393031}k_{10} - \frac{7740921184}{59489393031}k_{11}$$
$$+ \frac{1289681120}{19829797677}k_{12} - \frac{1935938912}{59489393031}k_{13} + \frac{966552224}{59489393031}k_{14} - \frac{53854816}{6609932559}k_{15}$$
$$+ \frac{240929440}{59489393031}k_{16} - \frac{121881952}{59489393031}k_{17} + \frac{19841248}{19829797677}k_{18} - \frac{31179104}{59489393031}k_{19}$$
$$+ \frac{14172320}{59489393031}k_{20} - \frac{2834464}{19829797677}k_{21} + \frac{2834464}{59489393031}k_{22} - \frac{2834464}{59489393031}k_{23}$$

$$c_4:=-\frac{143711027033024}{482525076807}k_0+\frac{191743328426296}{4342725691263}k_1-\frac{18506123050696}{4342725691263}k_2+\frac{490445635720}{1447575230421}k_3$$
$$-\frac{836069163824}{4342725691263}k_4+\frac{418010269280}{4342725691263}k_5-\frac{69650973760}{1447575230421}k_6+\frac{104375622016}{4342725691263}k_7$$
$$-\frac{51982545664}{4342725691263}k_8+\frac{2842703360}{482525076807}k_9-\frac{11974691840}{4342725691263}k_{10}+\frac{4355987456}{4342725691263}k_{11}$$
$$+\frac{362770432}{1447575230421}k_{12}-\frac{7073177600}{4342725691263}k_{13}+\frac{16598220800}{4342725691263}k_{14}-\frac{11472928768}{1447575230421}k_{15}$$
$$+\frac{69452333056}{4342725691263}k_{16}-\frac{139208458240}{4342725691263}k_{17}+\frac{1146388480}{17871299141}k_{18}-\frac{557218955264}{4342725691263}k_{19}$$
$$+\frac{1114478575616}{4342725691263}k_{20}-\frac{742991298560}{1447575230421}k_{21}+\frac{4457959751680}{4342725691263}k_{22}-\frac{8915921895424}{4342725691263}k_{23}$$
$$+\frac{16777216}{4085883}k_{24}$$

$$c_6:=\frac{25882619440443514917328}{3000823452662733}k_0-\frac{45853605500176015525552}{3000823452662733}k_1$$
$$+\frac{4019430638926241164336}{3000823452662733}k_2$$
$$+\frac{1310820135139425183376}{3000823452662733}k_3-\frac{1024455712527732368696}{3000823452662733}k_4$$
$$+\frac{512229321082704370224}{3000823452662733}k_5$$
$$-\frac{255994610360658257760}{3000823452662733}k_6+\frac{127761599274435484448}{3000823452662733}k_7-\frac{63523578732812331120}{3000823452662733}k_8$$
$$+\frac{31288912738841730088}{3000823452662733}k_9-\frac{15050064747426050024}{3000823452662733}k_{10}+\frac{6814985036721958572}{3000823452662733}k_{11}$$
$$-\frac{2575930203265077444}{3000823452662733}k_{12}+\frac{340747104191475552}{3000823452662733}k_{13}+\frac{898359358147981112}{3000823452662733}k_{14}$$
$$-\frac{1633568141058511104}{3000823452662733}k_{15}+\frac{2122687184107712322}{3000823452662733}k_{16}-\frac{2482901734955675584}{3000823452662733}k_{17}$$
$$+\frac{2784522617138716488}{3000823452662733}k_{18}-\frac{3050985997883844800}{3000823452662733}k_{19}+\frac{3305727115675958088}{3000823452662733}k_{20}$$
$$-\frac{3548742255546603220}{3000823452662733}k_{21}+\frac{3791742535543435884}{3000823452662733}k_{22}-\frac{4266024547552539200}{3000823452662733}k_{23}$$
$$+\frac{2409884745728}{8470035459}k_{24}$$

$$c_{12} := -\frac{2586970254909807601350 4}{3000823452662733}k_0 + \frac{458312257202643720997 6}{3000823452662733}k_1 - \frac{1339156154237336460 40}{1000274484220911}k_2$$

$$-\frac{1310180745806408609 68}{3000823452662733}k_3 + \frac{1023955965523411328 80}{3000823452662733}k_4 - \frac{1706598155631282147 2}{1000274484220911}k_5$$

$$+\frac{2558696735263629958 4}{3000823452662733}k_6 - \frac{1276991294373340428 8}{3000823452662733}k_7 + \frac{235156826366803712}{111141609357879}k_8$$

$$-\frac{3127329139682183680}{3000823452662733}k_9 + \frac{1504224915083465728}{3000823452662733}k_{10} - \frac{227035508086011904}{1000274484220911}k_{11}$$

$$+\frac{257394845918547968}{3000823452662733}k_{12} - \frac{33971039345106944}{3000823452662733}k_{13} - \frac{29965574874546176}{1000274484220911}k_{14}$$

$$+\frac{163405329522212864}{3000823452662733}k_{15} - \frac{212329003674533888}{3000823452662733}k_{16} + \frac{27599176189657088}{333424828073637}k_{17}$$

$$-\frac{278647792510173184}{3000823452662733}k_{18} + \frac{305485229036093440}{3000823452662733}k_{19} - \frac{110447846474088448}{1000274484220911}k_{20}$$

$$+\frac{356414880411926528}{3000823452662733}k_{21} - \frac{382254848374538240}{3000823452662733}k_{22} + \frac{144254500472258560}{1000274484220911}k_{23}$$

$$-\frac{2444663914496}{8470035459}k_{24}$$

$$r_1 := -\frac{702900184322271674368 0}{19829797677}k_0 + \frac{373581336731836576563 2}{59489393031}k_1 - \frac{3275364677288127365 12}{59489393031}k_2$$

$$-\frac{3557733428608082739 2}{19829797677}k_3 + \frac{8341384281975288627 2}{59489393031}k_4 - \frac{4169436997940906393 6}{59489393031}k_5$$

$$+\frac{2314310057092481024}{6609932559}k_6 - \frac{1038917306164832665 6}{59489393031}k_7 + \frac{5160109930397532160}{59489393031}k_8$$

$$-\frac{844788203802370048}{19829797677}k_9 + \frac{1206876838568296448}{59489393031}k_{10} - \frac{525583816383635456}{59489393031}k_{11}$$

$$+\frac{54337378444017664}{19829797677}k_{12} + \frac{44107571790061568}{59489393031}k_{13} - \frac{178852000356859904}{59489393031}k_{14}$$

$$+\frac{31365795610943488}{6609932559}k_{15} - \frac{376405569052377088}{59489393031}k_{16} + \frac{471713646880301056}{59489393031}k_{17}$$

$$-\frac{191639705395167232}{19829797677}k_{18} + \frac{688906000512671744}{59489393031}k_{19} - \frac{816558323406503936}{59489393031}k_{20}$$

$$+\frac{319292028288999424}{19829797677}k_{21} - \frac{1099193846327492608}{59489393031}k_{22} + \frac{1099193846327492608}{59489393031}k_{23}$$

$$r_2 := \frac{96870010362951904460 80}{19829797677}k_0 - \frac{5148501566898386541568}{59489393031}k_1 + \frac{4513834816192555571 20}{59489393031}k_2$$

$$+\frac{4903472893918498150 4}{19829797677}k_3 - \frac{11496636151705507840 0}{59489393031}k_4 + \frac{5746856965502712985 6}{59489393031}k_5$$

$$-\frac{3190110653564521472}{6609932559}k_6 + \frac{14322145792701082112}{59489393031}k_7 - \frac{7114017618187274240}{59489393031}k_8$$

$$+\frac{1164376425905979904}{19829797677}k_9 - \frac{1660594966200656896}{59489393031}k_{10} + \frac{717377775102722560}{59489393031}k_{11}$$

$$-\frac{70555693680738304}{19829797677}k_{12} - \frac{82153755554619904}{59489393031}k_{13} + \frac{279272462949016576}{59489393031}k_{14}$$

$$-\frac{48561934053684736}{6609932559}k_{15} + \frac{586623317616281600}{59489393031}k_{16} - \frac{743353203603327488}{59489393031}k_{17}$$

$$+\frac{305806752391226368}{19829797677}k_{18} - \frac{1113787856531140096}{59489393031}k_{19} + \frac{1337419379879514112}{59489393031}k_{20}$$

$$-\frac{529438275739600384}{19829797677}k_{21} + \frac{1839210274558088192}{59489393031}k_{22} - \frac{1839210274558088192}{59489393031}k_{23}$$

$$r_3 := -\frac{2224216433049605536 00}{2203310853}k_0 + \frac{1182114529261728505 12}{6609932559}k_1 - \frac{1036068072987679134 4}{6609932559}k_2$$

$$-\frac{1126886450090739824}{2203310853}k_3 + \frac{2641939308401506528}{6609932559}k_4 - \frac{1321060429928345728}{6609932559}k_5$$

$$+\frac{73364654855260832}{734436951}k_6 - \frac{329556839648748896}{6609932559}k_7 + \frac{163763828171344928}{6609932559}k_8$$

$$-\frac{26777244798632608}{2203310853}k_9 + \frac{37904965416550432}{6609932559}k_{10} - \frac{15788374880930272}{6609932559}k_{11}$$

$$+\frac{1183168629571936}{2203310853}k_{12} + \frac{4058051992105504}{6609932559}k_{13} - \frac{9752352542226400}{6609932559}k_{14}$$

$$+\frac{1658881916929312}{734436951}k_{15} - \frac{20389941940065248}{6609932559}k_{16} + \frac{26575465621977632}{6609932559}k_{17}$$

$$-\frac{11271646462038688}{2203310853}k_{18} + \frac{42322126391863840}{6609932559}k_{19} - \frac{52310789798604256}{6609932559}k_{20}$$

$$+\frac{21260309868779104}{2203310853}k_{21} - \frac{75251069414070368}{6609932559}k_{22} + \frac{75251069414070368}{6609932559}k_{23}$$

$$r_4 := -\frac{7392215223684602560}{19829797677}k_0 + \frac{3928551544346165912}{59489393031}k_1 - \frac{3440234843293732 88}{59489393031}k_2$$
$$-\frac{375432782576250 80}{19829797677}k_3 + \frac{88002219417588368}{59489393031}k_4 - \frac{44039308661585360}{59489393031}k_5$$
$$+\frac{2448412189382864}{6609932559}k_6 - \frac{11017980593474224}{59489393031}k_7 + \frac{5493635270869840}{59489393031}k_8$$
$$-\frac{905494935818128}{19829797677}k_9 + \frac{1312428808628048}{59489393031}k_{10} - \frac{595423007101616}{59489393031}k_{11}$$
$$+\frac{73813113073264}{19829797677}k_{12} - \frac{19469703165616}{59489393031}k_{13} - \frac{96995881980080}{59489393031}k_{14}$$
$$+\frac{18911830740688}{6609932559}k_{15} - \frac{222292541127856}{59489393031}k_{16} + \frac{263313375471952}{59489393031}k_{17}$$
$$-\frac{99768186587536}{19829797677}k_{18} + \frac{332277954021200}{59489393031}k_{19} - \frac{364245418269104}{59489393031}k_{20}$$
$$+\frac{131735650835440}{19829797677}k_{21} - \frac{426168486743536}{59489393031}k_{22} + \frac{426168486743536}{59489393031}k_{23}$$

$$r_5 := -\frac{605435005600528531456}{19829797677}k_0 + \frac{321791064440572780544}{59489393031}k_1 - \frac{282228324322057502 72}{59489393031}k_2$$
$$-\frac{3061544279114104832}{19829797677}k_3 + \frac{7178318207087722496}{59489393031}k_4 - \frac{3586610862513299456}{59489393031}k_5$$
$$+\frac{198970736961806336}{6609932559}k_6 - \frac{892576406743711744}{59489393031}k_7 + \frac{442988399109898240}{59489393031}k_8$$
$$-\frac{72494649067085824}{19829797677}k_9 + \frac{103736489461686272}{59489393031}k_{10} - \frac{45664975392456704}{59489393031}k_{11}$$
$$+\frac{5048883154960384}{19829797677}k_{12} + \frac{1797635806060544}{59489393031}k_{13} - \frac{12239684442202112}{59489393031}k_{14}$$
$$+\frac{2181463130570752}{6609932559}k_{15} - \frac{25787153149984768}{59489393031}k_{16} + \frac{31523942159982592}{59489393031}k_{17}$$
$$-\frac{12445638960357376}{19829797677}k_{18} + \frac{43390537871900672}{59489393031}k_{19} - \frac{49849265691115520}{59489393031}k_{20}$$
$$+\frac{18904366779572224}{19829797677}k_{21} - \frac{63576934986317824}{59489393031}k_{22} + \frac{63576934986317824}{59489393031}k_{23}$$

$$r_6 := \frac{547724272121467328}{19829797677}k_0 - \frac{291109042431905144}{59489393031}k_1 + \frac{25524117437721224}{59489393031}k_2$$
$$+\frac{2771953718787352}{19829797677}k_3 - \frac{6498832879482320}{59489393031}k_4 + \frac{3247452732792656}{59489393031}k_5$$
$$-\frac{180040763813776}{6609932559}k_6 + \frac{804553420954096}{59489393031}k_7 - \frac{392865457928464}{59489393031}k_8$$
$$+\frac{60310626711632}{19829797677}k_9 - \frac{65167347394832}{59489393031}k_{10} - \frac{8111300299024}{59489393031}k_{11}$$
$$+\frac{22883185816144}{19829797677}k_{12} - \frac{135184827746576}{59489393031}k_{13} + \frac{222433933711088}{59489393031}k_{14}$$
$$-\frac{38271131214992}{6609932559}k_{15} + \frac{516581596636912}{59489393031}k_{16} - \frac{756240342511888}{59489393031}k_{17}$$
$$+\frac{361157375860304}{19829797677}k_{18} - \frac{1521006208191248}{59489393031}k_{19} + \frac{2091571840690160}{59489393031}k_{20}$$
$$-\frac{931723008359216}{19829797677}k_{21} + \frac{3498766209465136}{59489393031}k_{22} - \frac{3498766209465136}{59489393031}k_{23}$$

$$r_7 := \frac{7383201101818445824}{19829797677}k_0 - \frac{3923907825658625024}{59489393031}k_1 + \frac{343870742935448576}{59489393031}k_2$$
$$+\frac{37421170596667904}{19829797677}k_3 - \frac{87748189594738688}{59489393031}k_4 + \frac{43904678717227520}{59489393031}k_5$$
$$-\frac{2440827032511488}{6609932559}k_6 + \frac{10983846996928000}{59489393031}k_7 - \frac{5476568863221760}{59489393031}k_8$$
$$+\frac{02650404335104}{19829797677}k_9 - \frac{1308162402028544}{59489393031}k_{10} + \frac{593289413176832}{59489393031}k_{11}$$
$$-\frac{73457644294144}{19829797677}k_{12} + \frac{18936109371904}{59489393031}k_{13} + \frac{97262288251904}{59489393031}k_{14}$$
$$-\frac{18926674491904}{6609932559}k_{15} + \frac{222358947383296}{59489393031}k_{16} - \frac{263346969224704}{59489393031}k_7$$
$$+\frac{99773655337984}{19829797677}k_{18} - \frac{332286547771904}{59489393031}k_{19} + \frac{364249324519424}{59489393031}k_{20}$$
$$-\frac{131736432085504}{19829797677}k_{21} + \frac{426169267993600}{59489393031}k_{22} - \frac{426169267993600}{59489393031}k_{23}$$

$$r_8 := -\frac{27826021011840}{244812317}k_0 + \frac{14792578089776}{734436951}k_1 - \frac{1303810520912}{734436951}k_2$$

$$-\frac{136268844816}{244812317}k_3 + \frac{304125362720}{734436951}k_4 - \frac{115504514432}{734436951}k_5$$

$$-\frac{3816165600}{244812317}k_6 + \frac{136290345056}{734436951}k_7 - \frac{313610794784}{734436951}k_8$$

$$+\frac{205466416032}{244812317}k_9 - \frac{1164716789536}{734436951}k_{10} + \frac{2170047402208}{734436951}k_{11}$$

$$-\frac{1336571773536}{244812317}k_{12} + \frac{7352863856864}{734436951}k_{13} - \frac{13369694488352}{734436951}k_{14}$$

$$+\frac{8021956148640}{244812317}k_{15} - \frac{42783973041952}{734436951}k_{16} + \frac{74872052738272}{734436951}k_{17}$$

$$-\frac{42784046738016}{244812317}k_{18} + \frac{213920256365792}{734436951}k_{19} - \frac{342272422090016}{734436951}k_{20}$$

$$+\frac{171136212462240}{244812317}k_{21} - \frac{684544852683424}{734436951}k_{22} + \frac{684544852683424}{734436951}k_{23}$$

$$r_9 := -\frac{3561660864069632}{9565749}k_0 + \frac{1893025993080832}{28697247}k_1 - \frac{166027653971968}{28697247}k_2$$

$$-\frac{18007619743744}{9565749}k_3 + \frac{42211540025344}{28697247}k_4 - \frac{21076444499968}{28697247}k_5$$

$$+\frac{1167641755648}{3188583}k_6 - \frac{5225062387712}{28697247}k_7 + \frac{2583084744704}{28697247}k_8$$

$$-\frac{420738953216}{9565749}k_9 + \frac{601661980672}{28697247}k_{10} - \frac{271505477632}{28697247}k_{11}$$

$$+\frac{35435429888}{9565749}k_{12} - \frac{23827632128}{28697247}k_{13} - \frac{17532633088}{28697247}k_{14}$$

$$+\frac{4232425472}{3188583}k_{15} - \frac{48492363776}{28697247}k_{16} + \frac{53571694592}{28697247}k_{17}$$

$$-\frac{18744098816}{9565749}k_{18} + \frac{57441660928}{28697247}k_{19} - \frac{58167279616}{28697247}k_{20}$$

$$+\frac{19469717504}{9565749}k_{21} - \frac{58651025408}{28697247}k_{22} + \frac{58651025408}{28697247}k_{23}$$

$$r_{10}:=-\frac{476052335552}{19829797677}k_0+\frac{119032443832}{59489393031}k_1+\frac{7593272}{59489393031}k_2$$

$$-\frac{3280504}{19829797677}k_3+\frac{5592400}{59489393031}k_4-\frac{2796208}{59489393031}k_5$$

$$+\frac{155344}{6609932559}k_6-\frac{699056}{59489393031}k_7+\frac{349520}{59489393031}k_8$$

$$-\frac{58256}{19829797677}k_9+\frac{87376}{59489393031}k_{10}-\frac{43696}{59489393031}k_{11}$$

$$+\frac{7280}{19829797677}k_{12}-\frac{10928}{59489393031}k_{13}+\frac{5456}{59489393031}k_{14}$$

$$-\frac{304}{6609932559}k_{15}+\frac{1360}{59489393031}k_{16}-\frac{688}{59489393031}k_{17}$$

$$+\frac{112}{19829797677}k_{18}-\frac{176}{59489393031}k_{19}+\frac{80}{59489393031}k_{20}$$

$$-\frac{16}{19829797677}k_{21}+\frac{16}{59489393031}k_{22}-\frac{16}{59489393031}k_{23}$$

$$r_{11}:=\frac{6277794439232896}{55571902646499}k_0-\frac{1511160452025968}{55571902646499}k_1+\frac{653907099701008}{55571902646499}k_2$$

$$-\frac{467357095195472}{55571902646499}k_3+\frac{363862258780768}{55571902646499}k_4-\frac{281507794407584}{55571902646499}k_5$$

$$+\frac{232252137406528}{55571902646499}k_6-\frac{234086061531008}{55571902646499}k_7+\frac{313551521189632}{55571902646499}k_8$$

$$-\frac{508955664078848}{55571902646499}k_9+\frac{874239023509504}{55571902646499}k_{10}-\frac{1483285407727616}{55571902646499}k_{11}$$

$$+\frac{2434231789895680}{55571902646499}k_{12}-\frac{3853108962025472}{55571902646499}k_{13}+\frac{5896036638834688}{55571902646499}k_{14}$$

$$-\frac{8749330546294784}{55571902646499}k_{15}+\frac{12627146727227392}{55571902646499}k_{16}-\frac{17765913528664064}{55571902646499}k_{17}$$

$$+\frac{24412338952732672}{55571902646499}k_{18}-\frac{32799425358430208}{55571902646499}k_{19}+\frac{43132763818491904}{55571902646499}k_{20}$$

$$-\frac{55823536478388224}{55571902646499}k_{21}+\frac{73229177537953792}{55571902646499}k_{22}-\frac{108040459657084928}{55571902646499}k_{23}$$

$$+\frac{216080919314169856}{55571902646499}k_{24}$$

$$r_{12}:=\frac{839636025917302144}{500147123818491}k_0-\frac{706803993123002 72}{500147123818491}k_1-\frac{413579008055089 76}{500147123818491}k_2$$

$$+\frac{424484515230675 04}{500147123818491}k_3-\frac{325987687123138 88}{500147123818491}k_4+\frac{238525161170866 24}{500147123818491}k_5$$

$$-\frac{16599850764392384}{500147123818491}k_6+\frac{11322462955368448}{500147123818491}k_7-\frac{8592011928928256}{500147123818491}k_8$$

$$+\frac{9056553739899904}{500147123818491}k_9-\frac{13412523660535808}{500147123818491}k_{10}+\frac{22362888815767552}{500147123818491}k_{11}$$

$$-\frac{36566714550984704}{500147123818491}k_{12}+\frac{56590015831515136}{500147123818491}k_{13}-\frac{82869244035547136}{500147123818491}k_{14}$$

$$+\frac{115693941089124352}{500147123818491}k_{15}-\frac{155205456065626112}{500147123818491}k_{16}+\frac{201405514494558208}{500147123818491}k_{17}$$

$$-\frac{254184493088964608}{500147123818491}k_{18}+\frac{313377969200250880}{500147123818491}k_{19}-\frac{378711921456545792}{500147123818491}k_{20}$$

$$+\frac{449638307114106880}{500147123818491}k_{21}-\frac{531749559574200320}{500147123818491}k_{22}+\frac{695972064494387200}{500147123818491}k_{23}$$

$$-\frac{1391944128988774400}{500147123818491}k_{24}$$

$$r_{13}:=\frac{3445547247321011 40320}{500147123818491}k_0-\frac{1827669816700195 06540}{500147123818491}k_1+\frac{1324789528678741 55156}{500147123818491}k_2$$

$$-\frac{104255451888613760476}{500147123818491}k_3+\frac{81642352119103499192}{500147123818491}k_4-\frac{62561499307615989616}{500147123818491}k_5$$

$$+\frac{46813131046374981344}{500147123818491}k_6-\frac{34150617881300518336}{500147123818491}k_7+\frac{24281262242273581952}{500147123818491}k_8$$

$$-\frac{16873243349631512320}{500147123818491}k_9+\frac{11569954184799723008}{500147123818491}k_{10}-\frac{8011411261450544128}{500147123818491}k_{11}$$

$$+\frac{5860044486019119104}{500147123818491}k_{12}-\frac{4825796922806781952}{500147123818491}k_{13}+\frac{4684690919279538176}{500147123818491}k_{14}$$

$$-\frac{5287451915499857920}{500147123818491}k_{15}+\frac{6559650998291161088}{500147123818491}k_{16}-\frac{8496288321333753856}{500147123818491}k_{17}$$

$$+\frac{11145258257123667968}{500147123818491}k_{18}-\frac{14574279700890339328}{500147123818491}k_{19}+\frac{18890720593330012160}{500147123818491}k_{20}$$

$$-\frac{24309316815835174912}{500147123818491}k_{21}+\frac{31932223698471317504}{500147123818491}k_{22}-\frac{47178037463743602688}{500147123818491}k_{23}$$

$$+\frac{94356074927487205376}{500147123818491}k_{24}$$

$$r_{14}:=\frac{3068685778048}{941115051}k_0-\frac{777333490256}{941115051}k_1+\frac{399881434672}{941115051}k_2-\frac{379850317616}{941115051}k_3$$
$$+\frac{458466338464}{941115051}k_4-\frac{652525805600}{941115051}k_5+\frac{1050702670528}{941115051}k_6-\frac{1806921528320}{941115051}k_7$$
$$+\frac{3185531140096}{941115051}k_8-\frac{5634920543744}{941115051}k_9+\frac{9904635593728}{941115051}k_{10}$$
$$-\frac{17226073051136}{941115051}k_{11}+\frac{29580175532032}{941115051}k_{12}-\frac{50072204435456}{941115051}k_{13}$$
$$+\frac{83413590851584}{941115051}k_{14}-\frac{136457325461504}{941115051}k_{15}+\frac{218626270461952}{941115051}k_{16}$$
$$-\frac{341913159581696}{941115051}k_{17}+\frac{520023974281216}{941115051}k_{18}-\frac{766664696152064}{941115051}k_{19}$$
$$+\frac{1095541307047936}{941115051}k_{20}-\frac{1534065770676224}{941115051}k_{21}+\frac{2191885939769344}{941115051}k_{22}$$
$$-\frac{3507526277955584}{941115051}k_{23}+\frac{7015052555911168}{941115051}k_{24}$$

$$r_{15}:=-\frac{2582178341554060048417 6}{500147123818491}k_0+\frac{389574890096172811505 0}{500147123818491}k_1$$
$$-\frac{4394566246874311261 66}{500147123818491}k_2$$
$$+\frac{99561785637338845874}{500147123818491}k_3-\frac{82033507745153904484}{500147123818491}k_4+\frac{63978915183167375816}{500147123818491}k_5$$
$$-\frac{48166072162620385168}{500147123818491}k_6+\frac{35236440572231864096}{500147123818491}k_7-\frac{25113464685848925760}{500147123818491}k_8$$
$$+\frac{17504570622175862912}{500147123818491}k_9-\frac{12057940066915332352}{500147123818491}k_{10}+\frac{8413874715264728576}{500147123818491}k_{11}$$
$$-\frac{6228316684636595200}{500147123818491}k_{12}+\frac{5195867466260077568}{500147123818491}k_{13}-\frac{5066927195062687744}{500147123818491}k_{14}$$
$$+\frac{5655859011882146816}{500147123818491}k_{15}-\frac{6841649281359056896}{500147123818491}k_{16}+\frac{8564005192211907584}{500147123818491}k_{17}$$
$$-\frac{10810327122276751360}{500147123818491}k_{18}+\frac{13589653368586554368}{500147123818491}k_{19}-\frac{16954298066568110080}{500147123818491}k_{20}$$
$$+\frac{21008889487075005440}{500147123818491}k_{21}-\frac{26443374352632580096}{500147123818491}k_{22}+\frac{37312344083747729408}{500147123818491}k_{23}$$
$$-\frac{74624688167495458816}{500147123818491}k_{24}$$

$$r_{16} := -\frac{3104311892525352121160}{500147123818491}k_0 + \frac{4469440340549183739 2}{500147123818491}k_1 - \frac{3535577727645460144}{500147123818491}k_2$$

$$-\frac{1222669534885461 28}{500147123818491}k_3 + \frac{373263604058456 00}{500147123818491}k_4 - \frac{21318942050343904}{500147123818491}k_5$$

$$+\frac{14509573693584704}{500147123818491}k_6 - \frac{9215672733291520}{500147123818491}k_7 + \frac{5770016901625856}{500147123818491}k_8$$

$$-\frac{4475890089998848}{500147123818491}k_9 + \frac{5544247102567424}{500147123818491}k_{10} - \frac{9013069453453312}{500147123818491}k_{11}$$

$$+\frac{14658127056392192}{500147123818491}k_{12} - \frac{21911032402223104}{500147123818491}k_{13} + \frac{29802908743909376}{500147123818491}k_{14}$$

$$-\frac{36945955088220160}{500147123818491}k_{15} + \frac{41553113352077312}{500147123818491}k_{16} - \frac{41496248399577088}{500147123818491}k_{17}$$

$$+\frac{34441353840361472}{500147123818491}k_{18} - \frac{18118963626360832}{500147123818491}k_{19} - \frac{9611307605917696}{500147123818491}k_{20}$$

$$+\frac{53030230583459840}{500147123818491}k_{21} - \frac{127826457051529216}{500147123818491}k_{22} + \frac{277418909987667968}{500147123818491}k_{23}$$

$$-\frac{554837819975335936}{500147123818491}k_{24}$$

$$r_{17} := \frac{30874202617446521449856}{500147123818491}k_0 - \frac{5235394581769620446128}{500147123818491}k_1 + \frac{1102666601914986255440}{500147123818491}k_2$$

$$-\frac{614608074950533170544}{500147123818491}k_3 + \frac{510750235392985189088}{500147123818491}k_4 - \frac{411767515300075828672}{500147123818491}k_5$$

$$+\frac{321617828152632060800}{500147123818491}k_6 - \frac{243146061940053456640}{500147123818491}k_7 + \frac{178006228747246317056}{500147123818491}k_8$$

$$-\frac{126674585640269249536}{500147123818491}k_9 + \frac{88578738114253862912}{500147123818491}k_{10} - \frac{62344142081190842368}{500147123818491}k_{11}$$

$$+\frac{46122423626838597632}{500147123818491}k_{12} - \frac{37928449863864082432}{500147123818491}k_{13} + \frac{35900933769453535232}{500147123818491}k_{14}$$

$$-\frac{38439876383310536704}{500147123818491}k_{15} + \frac{44251190717318070272}{500147123818491}k_{16} - \frac{52359196079417958400}{500147123818491}k_{17}$$

$$+\frac{62054853225925050368}{500147123818491}k_{18} - \frac{72792234666155597824}{500147123818491}k_{19} + \frac{84351636415428657152}{500147123818491}k_{20}$$

$$-\frac{96293650504382341120}{500147123818491}k_{21} + \frac{109000889272697274368}{500147123818491}k_{22} - \frac{134415366809327140864}{500147123818491}k_{23}$$

$$+\frac{268830733618654281728}{500147123818491}k_{24}$$

$$r_{18}:=\frac{49888631248}{12257649}k_0-\frac{45976398458}{12257649}k_1+\frac{41835356038}{12257649}k_2-\frac{37655497730}{12257649}k_3$$

$$+\frac{33471553540}{12257649}k_4-\frac{29287609352}{12257649}k_5+\frac{25103665168}{12257649}k_6-\frac{20919720992}{12257649}k_7$$

$$+\frac{16735776832}{12257649}k_8-\frac{12551832704}{12257649}k_9+\frac{8367888640}{12257649}k_{10}-\frac{4183944704}{12257649}k_{11}$$

$$+\frac{1024}{12257649}k_{12}+\frac{4183942144}{12257649}k_{13}-\frac{8367884288}{12257649}k_{14}+\frac{12551824384}{12257649}k_{15}$$

$$-\frac{16735760384}{12257649}k_{16}+\frac{20919688192}{12257649}k_{17}-\frac{25103599616}{12257649}k_{18}+\frac{29287478272}{12257649}k_{19}$$

$$-\frac{33471291392}{12257649}k_{20}+\frac{37654973440}{12257649}k_{21}-\frac{41838393344}{12257649}k_{22}+\frac{50205233152}{12257649}k_{23}$$

$$-\frac{100410466304}{12257649}k_{24}$$

$$r_{19}:=-\frac{87197894138645360}{110318841}k_0+\frac{12652393592104750}{110318841}k_1-\frac{980878121481938}{110318841}k_2$$

$$-\frac{72229529033018}{110318841}k_3+\frac{36116907940468}{110318841}k_4-\frac{18058401913064}{110318841}k_5$$

$$+\frac{9027159830992}{110318841}k_6-\frac{4509342703520}{110318841}k_7+\frac{2248443860800}{110318841}k_8$$

$$-\frac{1115795931776}{110318841}k_9+\frac{547476845824}{110318841}k_{10}-\frac{261109110272}{110318841}k_{11}$$

$$+\frac{115910751232}{110318841}k_{12}-\frac{41064639488}{110318841}k_{13}+\frac{1549613056}{110318841}k_{14}$$

$$+\frac{20609790976}{110318841}k_{15}-\frac{34091380736}{110318841}k_{16}+\frac{43853900800}{110318841}k_{17}$$

$$-\frac{52376717312}{110318841}k_{18}+\frac{62139188224}{110318841}k_{19}-\frac{75620655104}{110318841}k_{20}$$

$$+\frac{97779801088}{110318841}k_{21}-\frac{137294305280}{110318841}k_{22}+\frac{216323313664}{110318841}k_{23}$$

$$-\frac{432646627328}{110318841}k_{24}$$

Formulas in terms of p and k:

$$E_{12}(q) = (1 + \frac{65\,520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n)$$

$$= (p^{24} + \frac{41052}{691}p^{23} + \frac{33980820}{691}p^{22} + \frac{1821930964}{691}p^{21} +$$

$$\frac{34092098862}{691}p^{20} + \frac{340884330852}{691}p^{19} + \frac{2158576080388}{691}p^{18} +$$

$$\frac{9452473799436}{691}p^{17} + \frac{30150062503677}{691}p^{16} + \frac{72378756074200}{691}p^{15} +$$

$$\frac{133558779300456}{691}p^{14} + \frac{191958602404488}{691}p^{13}$$

$$+ \frac{216461992062532}{691}p^{12} + \frac{191958602404488}{691}p^{11}$$

$$+ \frac{133558779300456}{691}p^{10} + \frac{72378756074200}{691}p^{9}$$

$$+ \frac{30150062503677}{691}p^{8} + \frac{9452473799436}{691}p^{7} + \frac{2158576080388}{691}p^{6}$$

$$+ \frac{340884330852}{691}p^{5} + \frac{34092098862}{691}p^{4} + \frac{1821930964}{691}p^{3}$$

$$+33980820/691p^2 + 41052/691p + 1)k^{12}$$

$$E_{12}(q^2)$$

$$= (p^{24} + 12p^{23} + \frac{57840}{691}p^{22} + \frac{286594}{691}p^{21} + \frac{9327867}{691}p^{20} +$$

$$\frac{85695402}{691}p^{19} + \frac{522489058}{691}p^{18} + \frac{2297328246}{691}p^{17} +$$

$$\frac{14731029909}{1382}p^{16} + \frac{17688435340}{691}p^{15} + \frac{32611712916}{691}p^{14} +$$

$$\frac{46855135428}{691}p^{13} + \frac{52835572642}{691}p^{12} + \frac{46855135428}{691}p^{11} +$$

$$\frac{32611712916}{691}p^{10} + \frac{17688435340}{691}p^{9} + \frac{14731029909}{1382}p^{8} +$$

$$\frac{2297328246}{691}p^{7} + \frac{522489058}{691}p^{6} + \frac{85695402}{691}p^{5} + \frac{9327867}{691}p^{4}$$

$$+ \frac{286594}{691}p^{3} + \frac{57840}{691}p^{2} + 12p + 1)k^{12}$$

$E_{12}(q^3)$

$$= (p^{24} + 12p^{23} + 60p^{22} + \frac{114604}{691}p^{21} + \frac{208302}{691}p^{20} + \frac{314652}{691}p^{19} + \frac{2514148}{691}p^{18} + \frac{19074996}{691}p^{17} + \frac{66904317}{691}p^{16} + \frac{134202280}{691}p^{15} + \frac{219067176}{691}p^{14} + \frac{361406328}{691}p^{13} + \frac{455600452}{691}p^{12} + \frac{361406328}{691}p^{11} + \frac{219067176}{691}p^{10} + \frac{134202280}{691}p^{9} + \frac{66904317}{691}p^{8} + \frac{19074996}{691}p^{7} + \frac{2514148}{691}p^{6} + \frac{314652}{691}p^{5} + \frac{208302}{691}p^{4} + \frac{114604}{691}p^{3} + 60p^2 + 12p + 1)k^{12}$$

$E_{12}(q^4)$

$$= (\frac{1}{4096}p^{24} - \frac{6117}{707584}p^{23} + \frac{8306835}{707584}p^{22} - \frac{67708781}{176896}p^{21} + \frac{451969689}{353792}p^{20} + \frac{646778847}{176896}p^{19} - \frac{259597793}{44224}p^{18} - \frac{1176345297}{88448}p^{17} + \frac{3854045619}{353792}p^{16} + \frac{1419710665}{44224}p^{15} + \frac{500562579}{44224}p^{14} - \frac{75228357}{11056}p^{13} + \frac{227283689}{22112}p^{12} + \frac{71290077}{2764}p^{11} + \frac{12691461}{691}p^{10} + \frac{8589095}{1382}p^{9} + \frac{11661777}{11056}p^{8} - \frac{370701}{2764}p^{7} - \frac{782873}{2764}p^{6} - \frac{12948}{691}p^{5} + \frac{126402}{691}p^{4} + 154p^3 + 60p^2 + 12p + 1)k^{12}$$

$E_{12}(q^6)$

$= (p^{24} + 12p^{23} + 60p^{22} + 154p^{21} + 177p^{20} - 78p^{19} - \frac{360542}{691}p^{18} -$

$\frac{409014}{691}p^{17} + \frac{37269}{1382}p^{16} + \frac{492340}{691}p^{15} + \frac{443316}{691}p^{14} -$

$\frac{18372}{691}p^{13} - \frac{279518}{691}p^{12} - \frac{18372}{691}p^{11} + \frac{443316}{691}p^{10} +$

$\frac{492340}{691}p^{9} + \frac{37269}{1382}p^{8} - \frac{409014}{691}p^{7} - \frac{360542}{691}p^{6} - 78p^5 +$

$177p^4 + 154p^3 + 60p^2 + 12p + 1)k^{12}$

$E_{12}(q^{12})$

$= (\frac{1}{4096}p^{24} + \frac{3}{1024}p^{23} + \frac{15}{1024}p^{22} + \frac{6139}{176896}p^{21} + \frac{4539}{353792}p^{20} -$

$\frac{26403}{176896}p^{19} + \frac{29939}{88448}p^{18} + \frac{557703}{88448}p^{17}$

$+ \frac{8545779}{353792}p^{16} + \frac{1456555}{44224}p^{15} - \frac{1836591}{44224}p^{14}$

$- \frac{2443827}{11056}p^{13} - \frac{6254161}{22112}p^{12} + \frac{267}{4}p^{11} + 606p^{10} + \frac{1265}{2}p^{9}$

$- \frac{333}{16}p^{8} - \frac{2421}{4}p^{7} - \frac{2093}{4}p^{6} - 78p^5 + 177p^4 + 154p^3 + 60p^2 + 12p + 1)k^{12}$

$$f_1 := \sum_{n=0}^{\infty} f_1(n)q^n = \frac{\eta^{20}(4z)\eta^{10}(6z)\eta^4(12z)}{\eta^{10}(2z)}$$

$= (-\frac{1}{32768}p^{21} - \frac{41}{65536}p^{20} - \frac{95}{16384}p^{19} - \frac{131}{4096}p^{18} - \frac{1901}{16384}p^{17}$

$- \frac{18839}{65536}p^{16} - \frac{15831}{32768}p^{15} - \frac{2079}{4096}p^{14} - \frac{423}{2048}p^{13}$

$+ \frac{559}{2048}p^{12} + \frac{575}{1024}p^{11} + \frac{125}{256}p^{10} + \frac{31}{128}p^{9} + \frac{17}{256}p^{8} + \frac{1}{128}p^{7})k^{12},$

$$f_2 := \sum_{n=0}^{\infty} f_2(n) q^n = \frac{\eta^{15}(4z)\eta^{15}(6z)\eta^{3}(12z)}{\eta^{9}(2z)}$$

$$= (-\frac{1}{8192}p^{20} - \frac{35}{16384}p^{19} - \frac{273}{16384}p^{18} - \frac{311}{4096}p^{17} - \frac{227}{1024}p^{16}$$

$$-\frac{6939}{16384}p^{15} - \frac{8217}{16384}p^{14} - \frac{2151}{8192}p^{13} + \frac{819}{4096}p^{12}$$

$$+\frac{1057}{2048}p^{11} + \frac{485}{1024}p^{10} + \frac{123}{512}p^{9} + \frac{17}{256}p^{8} + \frac{1}{128}p^{7})k^{12},$$

$$f_3 := \sum_{n=0}^{\infty} f_3(n) q^n = \frac{\eta^{10}(4z)\eta^{20}(6z)\eta^{2}(12z)}{\eta^{8}(2z)}$$

$$= (-\frac{1}{2048}p^{19} - \frac{29}{4096}p^{18} - \frac{23}{512}p^{17} - \frac{665}{4096}p^{16} - \frac{185}{512}p^{15}$$

$$-\frac{1991}{4096}p^{14} - \frac{159}{512}p^{13} + \frac{533}{4096}p^{12} + \frac{965}{2048}p^{11} + \frac{235}{512}p^{10}$$

$$+\frac{61}{256}p^{9} + \frac{17}{256}p^{8} + \frac{1}{128}p^{7})k^{12},$$

$$f_4 := \sum_{n=0}^{\infty} f_4(n) q^n = \eta^{5}(2z)\eta^{5}(4z)\eta^{13}(6z)\eta(12z)$$

$$= (-\frac{1}{128}p^{20} - \frac{25}{256}p^{19} - \frac{257}{512}p^{18} - \frac{1327}{1024}p^{17} - \frac{1441}{1024}p^{16}$$

$$+\frac{995}{1024}p^{15} + \frac{4807}{1024}p^{14} + \frac{4761}{1024}p^{13} - \frac{819}{1024}p^{12}$$

$$-\frac{5515}{1024}p^{11} - \frac{4097}{1024}p^{10} + \frac{17}{512}p^{9} + \frac{221}{128}p^{8} + \frac{35}{32}p^{7}$$

$$+\frac{19}{64}p^{6} + \frac{1}{32}p^{5})k^{12},$$

$$f_5 := \sum_{n=0}^{\infty} f_5(n) q^n = \frac{\eta^{16}(4z)\eta^{2}(6z)\eta^{8}(12z)}{\eta^{2}(2z)}$$

$$= (\frac{1}{16384}p^{22} + \frac{19}{16384}p^{21} + \frac{641}{65536}p^{20} + \frac{1565}{32768}p^{19}$$

$$+\frac{4801}{32768}p^{18} + \frac{2297}{8192}p^{17} + \frac{18505}{65536}p^{16} - \frac{1239}{32768}p^{15}$$

$$-\frac{2211}{4096}p^{14} - \frac{1475}{2048}p^{13} - \frac{689}{2048}p^{12} + \frac{199}{1024}p^{11} + \frac{95}{256}p^{10}$$

$$+\frac{29}{128}p^{9} + \frac{17}{256}p^{8} + \frac{1}{128}p^{7})k^{12},$$

$$f_6 := \sum_{n=0}^{\infty} f_6(n)q^n = \frac{\eta^9(2z)\eta^9(6z)\eta^9(12z)}{\eta^3(4z)}$$

$= (-1/128p^{20} - 17/256p^{19} - 105/512p^{18} - 215/1024p^{17} + 259/1024p^{16}$

$+819/1024p^{15} + 495/1024p^{14} - 495/1024p^{13} - 819/1024p^{12}$

$-259/1024p^{11} + 215/1024p^{10} + 105/512p^9 + 17/256p^8 + 1/128p^7)k^{12},$

$$f_7 := \sum_{n=0}^{\infty} f_7(n)q^n = \frac{\eta^{11}(2z)\eta^{11}(4z)\eta^7(12z)}{\eta^5(6z)}$$

$= (\frac{1}{1024}p^{23} + \frac{17}{1024}p^{22} + \frac{247}{2048}p^{21} + \frac{485}{1024}p^{20} + \frac{16361}{16384}p^{19}$

$+\frac{10703}{16384}p^{18} - \frac{8023}{4096}p^{17} - \frac{42317}{8192}p^{16} - \frac{56119}{16384}p^{15}$

$+\frac{71643}{16384}p^{14} + \frac{74321}{8192}p^{13} + \frac{3679}{1024}p^{12} - \frac{595}{128}p^{11}$

$-\frac{2851}{512}p^{10} - \frac{317}{256}p^9 + \frac{85}{64}p^8 + \frac{67}{64}p^7 + \frac{19}{64}p^6 + \frac{1}{32}p^5)k^{12},$

$$f_8 := \sum_{n=0}^{\infty} f_8(n)q^n = \frac{\eta^{12}(6z)\eta^{18}(12z)}{\eta^6(4z)}$$

$= (-\frac{1}{2048}p^{19} - \frac{13}{4096}p^{18} - \frac{1}{128}p^{17} - \frac{33}{4096}p^{16} + \frac{33}{4096}p^{14}$

$+\frac{1}{128}p^{13} + \frac{13}{4096}p^{12} + \frac{1}{2048}p^{11})k^{12},$

$$f_9 := \sum_{n=0}^{\infty} f_9(n)q^n = \frac{\eta^{14}(2z)\eta^8(4z)\eta^{16}(12z)}{\eta^{14}(6z)}$$

$= \frac{1}{4096}p^{24} + \frac{1}{256}p^{23} + \frac{215}{8192}p^{22} + \frac{755}{8192}p^{21} + \frac{10321}{65536}p^{20} + \frac{191}{32768}p^{19} -$

$\frac{16237}{32768}p^{18} - \frac{815}{1024}p^{17} - \frac{3959}{65536}p^{16} + \frac{37801}{32768}p^{15} +$

$\frac{4565}{4096}p^{14} - \frac{443}{2048}p^{13} - \frac{1937}{2048}p^{12} - \frac{457}{1024}p^{11} +$

$\frac{35}{256}p^{10} + \frac{25}{128}p^9 + \frac{17}{256}p^8 + \frac{1}{128}p^7)k^{12},$

$$f_{10} := \sum_{n=0}^{\infty} f_{10}(n)q^n = \frac{\eta^6(2z)\eta^{12}(4z)\eta^{18}(6z)}{\eta^{12}(12z)}$$

$$= (\frac{1}{16}p^{20} + p^{19} + \frac{219}{32}p^{18} + \frac{811}{32}p^{17} + \frac{12881}{256}p^{16} + \frac{3783}{128}p^{15}$$

$$-\frac{779}{8}p^{14} - \frac{3937}{16}p^{13} - \frac{21957}{128}p^{12} + \frac{11229}{64}p^{11} + \frac{26643}{64}p^{10}$$

$$+\frac{3603}{16}p^9 - \frac{38879}{256}p^8 - \frac{34777}{128}p^7 - \frac{7467}{64}p^6 + \frac{1091}{32}p^5$$

$$+\frac{931}{16}p^4 + \frac{213}{8}p^3 + \frac{23}{4}p^2 + \frac{1}{2}p)k^{12},$$

$$f_{11} := \sum_{n=0}^{\infty} f_{11}(n)q^n = \frac{\eta^{20}(2z)\eta^4(6z)\eta^4(12z)}{\eta^4(4z)}$$

$$= (-\frac{1}{8}p^{21} - \frac{17}{16}p^{20} - \frac{47}{16}p^{19} - \frac{21}{32}p^{18} + \frac{1443}{128}p^{17} + \frac{4275}{256}p^{16}$$

$$-\frac{1097}{128}p^{15} - \frac{4673}{128}p^{14} - \frac{1883}{128}p^{13} + \frac{1883}{64}p^{12} + \frac{3697}{128}p^{11}$$

$$-\frac{527}{128}p^{10} - \frac{531}{32}p^9 - \frac{1551}{256}p^8 + \frac{69}{32}p^7 + \frac{71}{32}p^6 + \frac{5}{8}p^5 + \frac{1}{16}p^4)k^{12},$$

$$f_{12} := \sum_{n=0}^{\infty} f_{12}(n)q^n = \frac{\eta^{19}(2z)\eta(4z)\eta^5(12z)}{\eta(6z)}$$

$$= (-\frac{1}{32}p^{22} - \frac{23}{64}p^{21} - \frac{25}{16}p^{20} - \frac{333}{128}p^{19} + \frac{935}{512}p^{18} + \frac{13277}{1024}p^{17}$$

$$+\frac{13173}{1024}p^{16} - \frac{14195}{1024}p^{15} - \frac{35731}{1024}p^{14} - \frac{9185}{1024}p^{13} + \frac{31643}{1024}p^{12} +$$

$$\frac{27007}{1024}p^{11} - \frac{5895}{1024}p^{10} - \frac{527}{32}p^9 - \frac{1445}{256}p^8 + \frac{147}{64}p^7 +$$

$$\frac{143}{64}p^6 + \frac{5}{8}p^5 + \frac{1}{16}p^4)k^{12},$$

$$f_{13}:=\sum_{n=0}^{\infty} f_{13}(n)q^n=\frac{\eta^{20}(4z)\eta^4(6z)\eta^4(12z)}{\eta^4(2z)}$$

$$=(\frac{1}{16384}p^{22}+\frac{21}{16384}p^{21}+\frac{793}{65536}p^{20}+\frac{1103}{16384}p^{19}+\frac{7931}{32768}p^{18}$$

$$+\frac{9395}{16384}p^{17}+\frac{55257}{65536}p^{16}+\frac{8633}{16384}p^{15}-\frac{10083}{16384}p^{14}$$

$$-\frac{1843}{1024}p^{13}-\frac{3639}{2048}p^{12}-\frac{245}{512}p^{11}+\frac{389}{512}p^{10}+\frac{31}{32}p^9$$

$$+\frac{133}{256}p^8+\frac{9}{64}p^7+\frac{1}{64}p^6)k^{12}.$$

$$f_{14}:=\sum_{n=0}^{\infty} f_{14}(n)q^n=\frac{\eta^{14}(6z)\eta^{14}(12z)}{\eta^2(2z)\eta^2(4z)}$$

$$=(-\frac{1}{2048}p^{19}-\frac{17}{4096}p^{18}-\frac{29}{2048}p^{17}-\frac{97}{4096}p^{16}-\frac{33}{2048}p^{15}$$

$$+\frac{33}{4096}p^{14}+\frac{49}{2048}p^{13}+\frac{77}{4096}p^{12}+\frac{7}{1024}p^{11}+\frac{1}{1024}p^{10})k^{12}.$$

$$f_{15}:=\sum_{n=0}^{\infty} f_{15}(n)q^n=\frac{\eta^{19}(4z)\eta^{17}(6z)}{\eta^{11}(2z)\eta(12z)}$$

$$=(-\frac{1}{8192}p^{20}-\frac{39}{16384}p^{19}-\frac{343}{16384}p^{18}-\frac{895}{8192}p^{17}-\frac{765}{2048}p^{16}$$

$$-\frac{14203}{16384}p^{15}-\frac{22095}{16384}p^{14}-\frac{81}{64}p^{13}-\frac{333}{1024}p^{12}$$

$$+\frac{469}{512}p^{11}+\frac{771}{512}p^{10}+\frac{19}{16}p^9+\frac{35}{64}p^8+\frac{9}{64}p^7+\frac{1}{64}p^6)k^{12}.$$

$$f_{16}:=\sum_{n=0}^{\infty} f_{16}(n)q^n=\frac{\eta^{18}(4z)\eta^{18}(6z)}{\eta^6(2z)\eta^6(12z)}$$

$$=(\frac{1}{1024}p^{20}+\frac{19}{1024}p^{19}+\frac{645}{4096}p^{18}+\frac{1599}{2048}p^{17}+\frac{10107}{4096}p^{16}$$

$$+\frac{1281}{256}p^{15}+\frac{23867}{4096}p^{14}+\frac{2587}{2048}p^{13}-\frac{32175}{4096}p^{12}$$

$$-\frac{3505}{256}p^{11}-\frac{2479}{256}p^{10}+\frac{3}{16}p^9+\frac{819}{128}p^8+\frac{93}{16}p^7+\frac{21}{8}p^6$$

$$+\frac{5}{8}p^5+\frac{1}{16}p^4)k^{12}.$$

$$f_{17}:=\sum_{n=0}^{\infty} f_{17}(n)q^n=\frac{\eta^{16}(4z)\eta^{8}(6z)\eta^{8}(12z)}{\eta^{8}(2z)}$$

$$=(-\frac{1}{32768}p^{21}-\frac{37}{65536}p^{20}-\frac{153}{32768}p^{19}-\frac{371}{16384}p^{18}-\frac{1159}{16384}p^{17}$$

$$-\frac{9567}{65536}p^{16}-\frac{783}{4096}p^{15}-\frac{513}{4096}p^{14}+\frac{45}{1024}p^{13}+\frac{379}{2048}p^{12}$$

$$+\frac{49}{256}p^{11}+\frac{27}{256}p^{10}+\frac{1}{32}p^{9}+\frac{1}{256}p^{8})k^{12}.$$

$$f_{18}:=\sum_{n=0}^{\infty} f_{18}(n)q^n=\frac{\eta^{12}(2z)\eta^{12}(4z)\eta^{12}(12z)}{\eta^{12}(6z)}$$

$$=(\frac{1}{4096}p^{24}+\frac{9}{2048}p^{23}+\frac{279}{8192}p^{22}+\frac{1185}{8192}p^{21}+\frac{22401}{65536}p^{20}$$

$$+\frac{657}{2048}p^{19}-\frac{15855}{32768}p^{18}-\frac{29277}{16384}p^{17}-\frac{108279}{65536}p^{16}$$

$$+\frac{16921}{16384}p^{15}+\frac{56061}{16384}p^{14}+\frac{2061}{1024}p^{13}-\frac{2823}{2048}p^{12}-\frac{1197}{512}p^{11}-$$

$$\frac{387}{512}p^{10}+\frac{15}{32}p^{9}+\frac{117}{256}p^{8}+\frac{9}{64}p^{7}+\frac{1}{64}p^{6})k^{12}.$$

$$f_{19}:=\sum_{n=0}^{\infty} f_{19}(n)q^n=\frac{\eta^{20}(4z)\eta^{16}(6z)\eta^{4}(12z)}{\eta^{16}(2z)}$$

$$=(\frac{1}{65536}p^{20}+\frac{5}{16384}p^{19}+\frac{91}{32768}p^{18}+\frac{249}{16384}p^{17}+\frac{3649}{65536}p^{16}$$

$$+\frac{589}{4096}p^{15}+\frac{1099}{4096}p^{14}+\frac{373}{1024}p^{13}+\frac{731}{2048}p^{12}+\frac{63}{256}p^{11}$$

$$+\frac{29}{256}p^{10}+\frac{1}{32}p^{9}+\frac{1}{256}p^{8})k^{12}.$$

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/9950