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# Witt Groups of $\mathbb{P}^1$

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## Abstract

In this paper we calculate the Witt groups of $\mathbb{P}^1$. It's a known result, but we calculate it by another method: we use the localisation theorem of Balmer and the excision theorem of S. Gille.

*Keywords: Witt group; toric variety; line bundle; filtration; divisor.*

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## 1 Background

### 1.1 Witt Groups of a Shifted and Twisted Scheme

Let $X$ be a scheme which contains $\frac{1}{2}$ and $VB_X$ be the category of locally free coherent $\mathcal{O}_X$-modules, i.e. vector bundles. Let $\mathcal{L}$ be a line bundle over $X$. We define a duality

$$\begin{array}{rcl} * : VB_X & \longrightarrow & VB_X \\ \mathcal{E} & \longmapsto & *(\mathcal{E}) := \mathcal{E}^* = Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \end{array}$$

which is the usual duality twisted by the line bundle $\mathcal{L}$. We identify naturally $\varpi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^{**}$. If $\mathcal{L} = \mathcal{O}_X$, then $\mathcal{E}^*$ is the usual dual and $\varpi$ is locally given by the application that maps an element $e$ of $\mathcal{E}$ to the evaluation at $e$. The triple $(VB_X, *, \varpi)$ is an exact category with duality.

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**Definition 1.1.** The Witt group of a scheme $X$ twisted by the line bundle $\mathcal{L}$ is:

$$\mathcal{W}(X,\mathcal{L}) := \mathcal{W}(VB_X,*,\varpi) \tag{1.1}$$

For the particular case $\mathcal{L} = \mathcal{O}_X$, we denote $\mathcal{W}(X,\mathcal{L}) = \mathcal{W}(X)$.

## 1.2 Derived Witt Group

Let $\mathcal{D}^b(VB_X)$ be the derived category of bounded complexes of vector bundles. We provide this category by a twisted shifted duality which is composed by a duality functor $\varpi_{n,\mathcal{E}^\cdot} : \mathcal{E}^\cdot \longrightarrow D_{L[n]}D_{L[n]}(\mathcal{E}^\cdot)$ and functorial isomorphisms of biduality $D_{L[n]} : \mathcal{E}^\cdot \longrightarrow \mathcal{E}^{\cdot\vee} \otimes L[n]$.

We represent the derived Witt group by:

$$\mathcal{W}^n(X,L) := \mathcal{W}\big(\mathcal{D}^b(VB_X), D_{L[n]}, 1, \varpi_{n,\bullet}\big).$$

Elements of $\mathcal{W}^n(X,L)$ are isometric classes of such $(\mathcal{E}^\cdot,\phi^\cdot)$ with

$$\phi^\cdot : \mathcal{E}^\cdot \longrightarrow D_{L[n]}(\mathcal{E}^\cdot)$$

is a symmetric isomorphism, with addition

$$[\mathcal{E}^\cdot,\phi^\cdot] + [\mathcal{F}^\cdot,\psi^\cdot] = [\mathcal{E}^\cdot \oplus \mathcal{F}^\cdot, \left(\begin{smallmatrix}\varphi^\cdot & 0\\ 0 & \psi^\cdot\end{smallmatrix}\right)]$$

modulo metabolic classes, and the opposite is

$$-[\mathcal{E}^\cdot,\phi^\cdot] = [\mathcal{E}^\cdot,-\phi^\cdot].$$

Witt groups are functorial. To a morphism $f : Y \to X$, we have pullbacks

$$\begin{array}{rcl} f^* : \mathcal{W}^n(X,L) & \longrightarrow & \mathcal{W}^n(Y,f^*L) \\ \left[\mathcal{E}^\cdot,\phi^\cdot\right] & \longmapsto & \left[f^*\mathcal{E}^\cdot, f^*\phi^\cdot\right] \end{array}$$

We have also a multiplication (Gille-Nenashev)

$$\begin{array}{rcl} \mathcal{W}^n(X,L_1)\times\mathcal{W}^m(X,L_2) & \longrightarrow & \mathcal{W}^{n+m}(X,L_1\otimes L_2) \\ \Big([\mathcal{E}^\cdot,\phi^\cdot],[\mathcal{F}^\cdot,\psi^\cdot]\Big) & \longmapsto & [\mathcal{E}^\cdot\otimes\mathcal{F}^\cdot,\phi^\cdot\otimes\psi^\cdot] \end{array}$$

This product is anticommutative:

$$[\mathcal{E}^\cdot\otimes\mathcal{F}^\cdot,\phi^\cdot\otimes\psi^\cdot] = (-1)^{nm}[\mathcal{F}^\cdot\otimes\mathcal{E}^\cdot,\psi^\cdot\otimes\phi^\cdot].$$

**Theorem 1.1.** *(Homotopic Invariance [**Balmer**])*

*Let $\pi : X\times\mathbb{A}^1 \longrightarrow X$ be the projection, and $i : X \longrightarrow X\times\mathbb{A}^1$ the section $x \mapsto (x,0)$. Then $\pi^*$ and $i^*$ are inverse isomorphisms:*

$$\mathcal{W}^n(X,L) \underset{i^*}{\overset{\pi^*}{\rightleftarrows}} \mathcal{W}^n(X\times\mathbb{A}^1,\pi^*L) \tag{1.2}$$

*Proof.* See [1]. □

To a closed subset $Z\subset X$, there is a subcategory $\mathcal{D}^b_Z(VB_X)\subset\mathcal{D}^b(VB_X)$ of bounded complexes of vector bundles over $X$ which are exact over $U = X\setminus Z$. The Witt groups of this subcategory are denoted $\mathcal{W}^n_Z(X,L)$.

**Theorem 1.2.** *(Localization [**Balmer**])*

*There is a long sequence*

$$\cdots \to \mathcal{W}^n_Z(X,L) \xrightarrow{Inclusion} \mathcal{W}^n(X,L) \xrightarrow{Restriction} \mathcal{W}^n(U,L|_U) \xrightarrow{\partial} \mathcal{W}^{n+1}_Z(X,L) \to \cdots \quad (1.3)$$

*when $\partial$ is explicit. To a class in $\mathcal{W}^n(U,L|_U)$, we can write $[\mathcal{E}^{\cdot}_U, \phi^{\cdot}_U]$ when $\phi^{\cdot} : \mathcal{E}^{\cdot} \to D_{L[n]}(\mathcal{E}^{\cdot})$ is a symmetric morphism of $\mathcal{D}(VB_X)$ such that its restriction over $U$ is an isomorphism. The mapping cone $C(\phi^{\cdot})$ is exact over $U$ and belongs to the subcategory $\mathcal{D}_Z(VB_X)$. Balmer provides $C(\phi^{\cdot})$ with a symmetric isomorphism $\psi^{\cdot} : C(\phi^{\cdot}) \to D_{L[n+1]}(C(\phi^{\cdot}))$ which is unique up to an isometry, and we set $\partial([\mathcal{E}^{\cdot}, \phi^{\cdot}]) = [C(\phi^{\cdot}), \psi^{\cdot}]$.*

*Proof.* See [2]. □

**Theorem 1.3.** *(Excision [**Gille**])*

*If $i : Z \hookrightarrow X$ is the inclusion of a closed subset $Z \subset X$ with codimension $d$, where $Z$ and $X$ are smooth, then there is a natural isomorphism*

$$i_* : \mathcal{W}^n(Z, L|_Z \otimes detN_{Z/X}) \xrightarrow{\simeq} \mathcal{W}^{n+d}_Z(X,L). \quad (1.4)$$

*Proof.* See [3]. □

If $i$ is the inclusion $i : Z \hookrightarrow Z \times \mathbb{A}^d$ given by $i(z) = (z, 0)$, then it may be explicit. Suppose that $x_1, x_2, \cdots, x_d$ are the standard coordinates in $\mathbb{A}^d$, and $K^{\cdot}(x_1, \cdots, x_d)$ is the *Koszul* complex.

**Theorem 1.4.** *Consider the inclusion $i : Z \hookrightarrow Z \times \mathbb{A}^d$ and denote the projection $\pi : Z \times \mathbb{A}^d \to Z$. The isomorphism of the excision theorem is:*

$$\begin{array}{rcl} i_* : \mathcal{W}^n(Z, \mathcal{L}_{|Z}) & \longrightarrow & \mathcal{W}^{n+d}_Z(Z \times \mathbb{A}^d, \mathcal{L}) \\ \left[\mathcal{E}^{\cdot}, \phi^{\cdot}\right] & \longmapsto & \left[(\pi^*\mathcal{E}, \pi^*\phi) \otimes K^{\cdot}(x_1, \cdots, x_d), k^{\cdot}\right] \end{array}$$

*where $k^{\cdot}$ is a symmetric isomorphism between $\left[(\pi^*\mathcal{E}, \pi^*\phi) \otimes K^{\cdot}(x_1, \cdots, x_d), k^{\cdot}\right]$ and its shifted dual.*

*Proof.* See [3]. □

**Theorem 1.5 (Balmer).** *The Witt groups of a point $\boldsymbol{Spec}(k) = \mathbb{A}^0_k = \mathbb{P}^0_k = * = pt$ are*

$$\mathcal{W}^n(\mathbb{A}^0_k, \mathcal{O}) = \begin{cases} \mathcal{W}(k) & \text{for } n \equiv 0 \ (mod\, 4) \\ 0 & \text{otherwise} \end{cases} \quad (1.5)$$

*where $\mathcal{W}(k)$ denotes the Witt group of isometry classes of anisotropic quadratic forms over $k$.*

*Proof.* See [4]. □

*Remark* 1.1. In this work, the value of $\mathcal{W}(k)$ is not important.

**Theorem 1.6 (Walter).** *Let $X$ be a scheme which contains $\frac{1}{2}$ . Consider the projective space $\mathbb{P}^r_X$ over $X$ such that $r \geq 1$. Let $m \in \mathbb{Z}/2$ and $\mathcal{O}(m) \in$ **Pic** $(\mathbb{P}^r_X)/2$.*

$$\text{If } \boldsymbol{r} \text{ is even, then } \mathcal{W}^i\big(\mathbb{P}^r_X, \mathcal{O}(m)\big) = \begin{cases} \mathcal{W}^i(X) & \text{if } m \text{ is even} \\ \mathcal{W}^{i-r}(X) & \text{if } m \text{ is odd} \end{cases}$$

$$\text{If } \boldsymbol{r} \text{ is odd, then } \mathcal{W}^i\big(\mathbb{P}^r_X, \mathcal{O}(m)\big) = \begin{cases} \mathcal{W}^i(X) \oplus \mathcal{W}^{i-r}(X) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

*Proof.* See [5]. □

## 1.3 Torus

Let $\mathbb{G}_m = \mathbb{A}^1 \setminus 0$. This $\mathbb{G}_m$ is an affine variety: $\mathbb{G}_m = \mathbf{Spec}(k[T, T^{-1}])$.

**Definition 1.2.** An algebraic torus is an algebraic group which is isomorphic to a finite product of $\mathbb{G}_m$:

$$\mathbb{G}_m \times \mathbb{G}_m ... \times \mathbb{G}_m = \mathbb{G}_m^n.$$

**Theorem 1.7.** *Let $x$ be the coordinate on $\mathbb{G}_m$. For all variety $Y$, all line bundle $\mathcal{L}$ over $Y$ and all $n$ we have the isomorphism:*

$$\begin{array}{ccc} \mathcal{W}^n(Y,\mathcal{L}) \oplus \mathcal{W}^n(Y,\mathcal{L}) & \stackrel{\simeq}{\longrightarrow} & \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^*\mathcal{L}) \\ ([\mathcal{E}^{\cdot}, \phi^{\cdot}], [\mathcal{F}^{\cdot}, \psi^{\cdot}]) & \mapsto & \left[\pi^*\mathcal{E}^{\cdot} \oplus \pi^*\mathcal{F}^{\cdot}, \left(\begin{smallmatrix} \pi^*\varphi^{\cdot} & 0 \\ 0 & x\pi^*\psi^{\cdot} \end{smallmatrix}\right)\right]. \end{array}$$

*We can denote that isomorphism by $(1, \langle x \rangle) : (e, f) \mapsto e + \langle x \rangle f$, when we identify every symmetric complex in $Y$ to its pullback into $\mathcal{W}^n(Y \times \mathbb{G}_m)$.*

*Proof.* See [3]. □

*Remark* 1.2. We have a long localisation exact sequence:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathcal{W}^n_{s_0(Y)}(Y \times \mathbb{A}^1, \pi^*\mathcal{L}) & \longrightarrow & \mathcal{W}^n(Y \times \mathbb{A}^1, \pi^*\mathcal{L}) & \xrightarrow{j^*} & \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^*\mathcal{L}) & \xrightarrow{\partial} & \mathcal{W}^{n+1}_{s_0(Y)}(Y \times \mathbb{A}^1, \pi^*\mathcal{L}) & \longrightarrow \cdots \\ & & & \pi^* \uparrow \cong & \swarrow s_1^* & & \nwarrow \langle x \rangle & s_{0*} \uparrow \cong & \\ & & & \mathcal{W}^n(Y,\mathcal{L}) & & & & \mathcal{W}^n(Y,\mathcal{L}) & \end{array}$$

Where $s_0 : Y \hookrightarrow Y \times \mathbb{A}^1$ is the null section and $s_1 : Y \hookrightarrow Y \times \mathbb{A}^1$ is the constant section at 1.

**Lemma 1.8.** *There is an isomorphism between the localisation exact sequence and the following one:*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{W}^n(Y \times \mathbb{A}^1, \pi^*\mathcal{L}) & \xrightarrow{j^*} & \mathcal{W}^n(Y \times \mathbb{G}_m, \pi^*\mathcal{L}) & \xrightarrow{\partial} & \mathcal{W}^{n+1}_{s_0(Y)}(Y \times \mathbb{A}^1, \pi^*\mathcal{L}) & \longrightarrow 0 \\ & \pi^* \uparrow \cong & & (\pi^*, \langle x \rangle.\pi^*) \uparrow \cong & & s_{0*} \uparrow \cong & \\ 0 \longrightarrow & \mathcal{W}^n(Y,\mathcal{L}) & \xrightarrow{i_1} & \mathcal{W}^n(Y,\mathcal{L}) \oplus \mathcal{W}^n(Y,\mathcal{L}) & \xrightarrow{p_2} & \mathcal{W}^n(Y,\mathcal{L}) & \longrightarrow 0 \end{array}$$

*where $i_1$ and $p_2$ denote the inclusion of the first factor and the projection on the second one, $s_0$ the null section and finally $x$ is the coordinate on $\mathbb{A}^1$ which vanishes at $0$.*

*Proof.* See [3]. □

*Remark* 1.3. The Witt groups of $\mathbb{G}_m$ are known; if $x_1, x_2, \cdots, x_n$ are the coordinates on $\mathbb{G}_m^n$, then

$$\mathcal{W}^1(\mathbb{G}_m^n) = \mathcal{W}^2(\mathbb{G}_m^n) = \mathcal{W}^3(\mathbb{G}_m^n) = 0.$$

Also we have:

$$\mathcal{W}^0(\mathbb{G}_m) = \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x \rangle,$$

and

$$\mathcal{W}^0(\mathbb{G}_m \times \mathbb{G}_m) = \mathcal{W}(k)\langle 1 \rangle \oplus \mathcal{W}(k)\langle x_1 \rangle \oplus \mathcal{W}(k)\langle x_2 \rangle \oplus \mathcal{W}(k)\langle x_1 x_2 \rangle.$$

etc.

# 2 Witt Groups of $\mathbb{P}^1$

Let $k$ an algebraically closed field and $\mathbb{P}^1 := \mathbb{P}^1_k$. Let $\mathcal{D}^b(\mathbb{P}^1) = \mathcal{D}^b(VB_{\mathbb{P}^1})$ the derived category of bounded complexes of vector bundles over $\mathbb{P}^1$ with the usual duality $\mathcal{E}^{\bullet\vee} = Hom_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}^\bullet, \mathcal{O}_{\mathbb{P}^1})$.

Let calculate the Witt groups of $\mathbb{P}^1$ using the localisation sequence with the closed subset $Z = \{0\} \cup \{\infty\}$ and its open complementary $\mathbb{G}_m$. Firstly we have $\mathbf{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$. As Witt groups are periodic modulo 2 on $L \in \mathbf{Pic}(X)$, so it really remains two kinds of groups to calculate: $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$.

## 2.1 Calculation of $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

**Theorem 2.1.** *For all $n \in \mathbb{N}$,*

$$W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \begin{cases} W(k) & \text{if } n \equiv 0 \text{ or } 1 \ [4], \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

*Proof.* We have the following exact sequence:

$$\begin{array}{ccccccccc} \cdots \longrightarrow \mathcal{W}^n(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^n(\mathbb{G}_m) & \xrightarrow{\partial} & \mathcal{W}^{n+1}_{0,\infty}(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^{n+1}(\mathbb{P}^1) \longrightarrow \cdots . \\ & & {\scriptstyle(\langle 1\rangle, \langle x\rangle)}\uparrow\cong & & \cong\uparrow{\scriptstyle(i_{0*}, i_{\infty*})} & & \\ & & \mathcal{W}^n(k) \oplus \mathcal{W}^n(k) & & \mathcal{W}^n(k) \oplus \mathcal{W}^n(k) & & \end{array}$$

As $\mathcal{W}^n(k) = 0$ for $n \neq 0$ $(\texttt{mod}4)$, we found $\mathcal{W}^2(\mathbb{P}^1) = 0$ and $\mathcal{W}^3(\mathbb{P}^1) = 0$, and it becomes the exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathcal{W}^0(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^0(\mathbb{G}_m) & \xrightarrow{(\partial_0, \partial_\infty)} & \mathcal{W}^1_0(\mathbb{P}^1) \oplus \mathcal{W}^1_\infty(\mathbb{P}^1) & \longrightarrow & \mathcal{W}^1(\mathbb{P}^1) \longrightarrow 0 . \\ & & {\scriptstyle(\langle 1\rangle, \langle x\rangle)}\uparrow\cong & & \cong\uparrow{\scriptstyle(i_{0*}, i_{\infty*})} & & \\ & & \mathcal{W}(k) \oplus \mathcal{W}(k) & & \mathcal{W}(k) \oplus \mathcal{W}(k) & & \end{array}$$

We can separate two connected components 0 and $\infty$.

Then we obtains

$$\partial_0(a\langle 1\rangle + b\langle x\rangle) = i_{0*}(b)$$

and

$$\partial_\infty(a\langle 1\rangle + b\langle x\rangle) = \partial_\infty(a\langle 1\rangle + b\langle x^{-1}\rangle) = i_{\infty*}(b)$$

because $\langle x\rangle = \langle x^{-1}\rangle$.

Thus it grows

$$0 \to \mathcal{W}^0(\mathbb{P}^1) \to \mathcal{W}(k)\langle 1\rangle \oplus \mathcal{W}(k)\langle x\rangle \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \mathcal{W}(k) \oplus \mathcal{W}(k) \to \mathcal{W}^1(\mathbb{P}^1) \to 0.$$

We define a filtration of $\mathcal{D}^b(\mathbb{P}^1)$ as

$$0 \subseteq \mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1) \subseteq \mathcal{D}^b(\mathbb{P}^1).$$

That gives us a short exact sequence of categories:

$$0 \to \mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1) \hookrightarrow \mathcal{D}^b(\mathbb{P}^1) \twoheadrightarrow \mathcal{D}^b(\mathbb{P}^1)/\mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1) \to 0.$$

Where

$$\mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1) \cong \mathcal{D}^b_{\{0\}}(\mathbb{P}^1) \amalg \mathcal{D}^b_{\{\infty\}}(\mathbb{P}^1)$$

and

$$\mathcal{D}^b(\mathbb{P}^1)/\mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1) \cong \mathcal{D}^b(\mathbb{P}^1 \setminus \{0,\infty\}).$$

Now

$$\mathcal{W}^p_{\{0,\infty\}}(\mathbb{P}^1) \cong \mathcal{W}^p_{\{0\}}(\mathbb{P}^1) \oplus \mathcal{W}^p_{\{\infty\}}(\mathbb{P}^1).$$

Then with respect to *the excision theorem* of **Gille**, we obtain:

$$\mathcal{W}^p_{\{0\}}(\mathbb{P}^1) := \mathcal{W}^p(\mathcal{D}^b_{\{0\}}(\mathbb{P}^1)) \cong \mathcal{W}^{p-1}(\{0\}),$$

and

$$\mathcal{W}^p_{\{\infty\}}(\mathbb{P}^1) := \mathcal{W}^p(\mathcal{D}^b_{\{\infty\}}(\mathbb{P}^1)) \cong \mathcal{W}^{p-1}(\{\infty\}).$$

Thus, if $p \equiv 1\ (\mathtt{mod}4)$, we have

$$\mathcal{W}^p_{\{0\}}(\mathbb{P}^1) \cong \mathcal{W}(k) \quad et \quad \mathcal{W}^p_{\{\infty\}}(\mathbb{P}^1) \cong \mathcal{W}(k).$$

Recall that for $x = \frac{X_0}{X_1}$ where $X_0 = 0$ at $\{0\}$ and $X_1 = 0$ at $\{\infty\}$, the isomorphism $\mathcal{W}(k) \cong \mathcal{W}^p_{\{0\}}(\mathbb{P}^1)$ is described by:

$$\langle a_1, a_2, ..., a_r\rangle \mapsto \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r} & \xrightarrow{\begin{pmatrix} X_0 & 0 & \cdot & 0 \\ 0 & X_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & X_0 \end{pmatrix}} & \mathcal{O}_{\mathbb{P}^1}{}^{\oplus r} & \longrightarrow & 0 \\ & & \Big\downarrow \begin{pmatrix} -a_1X_1 & 0 & \cdot & 0 \\ 0 & -a_2X_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & -a_rX_1 \end{pmatrix} & & \Big\downarrow \begin{pmatrix} a_1X_1 & 0 & \cdot & 0 \\ 0 & a_2X_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & a_rX_1 \end{pmatrix} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}{}^{\oplus r} & \xrightarrow[\begin{pmatrix} -X_0 & 0 & \cdot & 0 \\ 0 & -X_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & -X_0 \end{pmatrix}]{} & \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} & \longrightarrow & 0 \end{array}$$

With respect to *the localisation theorem* of **Balmer**, the spectral sequence is reduced to:

$$\cdots \to \mathcal{W}^p(\mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1)) \xrightarrow{\alpha} \mathcal{W}^p(\mathcal{D}^b(\mathbb{P}^1)) \xrightarrow{\beta} \mathcal{W}^p(\mathcal{D}^b(\mathbb{P}^1 \setminus \{0,\infty\})) \xrightarrow{\partial} \mathcal{W}^{p+1}(\mathcal{D}^b_{\{0,\infty\}}(\mathbb{P}^1)) \to \cdots,$$

where $\alpha$ is the inclusion and $\beta$ is the restriction.

Then for $p = 0$, we have:

$$0 \to \mathcal{W}^0(\mathbb{P}^1) \to \mathcal{W}^0(\mathbb{G}_m) \xrightarrow{\partial} \mathcal{W}^1_{\{0,\infty\}}(\mathbb{P}^1) \to \mathcal{W}^1(\mathbb{P}^1) \to 0.$$

Recall that $\mathbb{G}_m = Spec(k[t,t^{-1}])$ and $\mathcal{W}^0(\mathbb{G}_m) \cong \mathcal{W}(k)\langle 1\rangle \oplus \mathcal{W}(k)\langle x\rangle$ which is a free $\mathcal{W}(k)$-module of *rank* 2.

Describe now $\partial(\langle 1\rangle)$ and $\partial(\langle x\rangle)$.

- $$\langle 1\rangle := \begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \\ & & \Big\downarrow 1 & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \end{array} \quad \text{and} \quad \partial(\langle 1\rangle) = \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{1} & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \\ & & 0 \Big\downarrow & & \Big\downarrow 0 & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{-1} & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \end{array}$$

The two lines of $\partial(\langle 1\rangle)$ are acyclic complexes so $\partial(\langle 1\rangle) = 0$, then

$$\mathcal{W}(k)\langle 1\rangle \subset \mathbf{ker}(\partial) = \mathcal{W}^0(\mathbb{P}^1)$$

$$\langle x\rangle := \begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1}(-1) & \longrightarrow & 0 \\ & & \downarrow X_0 & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \end{array} \quad \text{and} \quad \partial(\langle x\rangle) = \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{X_0} & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \\ & & {\scriptstyle -X_1}\downarrow & & \downarrow{\scriptstyle X_1} & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{-X_0} & \mathcal{O}_{\mathbb{P}^1} & \longrightarrow & 0 \end{array}$$

which prove that $\partial(\langle x\rangle) = \langle 1\rangle$.

Then $\langle 1\rangle \longmapsto (0,0)$ and $\langle x\rangle \longmapsto (\langle 1\rangle, \langle 1\rangle)$.

Next, $\mathbb{P}^1$ with trivial duality has the following Witt groups:

$$\mathcal{W}^0(\mathbb{P}^1) = \mathbf{ker}(\partial) = \mathcal{W}(k)\langle 1\rangle,$$

and

$$\mathcal{W}^1(\mathbb{P}^1) = \mathbf{coker}(\partial) = \frac{\mathcal{W}(k) \oplus \mathcal{W}(k)}{\mathcal{W}(k)(\langle 1\rangle, \langle 1\rangle)} \cong \mathcal{W}(k).$$

□

## 2.2 Calculation of $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$

**Theorem 2.2.** *For all $n \in \mathbb{N}$, $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$.*

The groups $\mathcal{W}^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ are more complicated. We use the theory of divisors.

**Definition 2.1.** An irreducible divisor on a smooth variety X is an irreducible subvariety $Z \subset X$ of codimension 1. A divisor on a smooth variety $X$ is a formal sum of irreducible divisors with coefficients in $\mathbb{Z}$

$$D = a_1 Z_1 + a_2 Z_2 + \cdots + a_r Z_r.$$

Divisors on X form an abelian group $\mathbf{Div}(X)$. A divisor is effective if all its coefficients $a_i \geq 0$. We write $D \succ E$ if $D - E$ E is effective.

For an open $U \subset X$, we have a restriction morphism

$$\begin{array}{ccc} \mathbf{Div}(X) & \longrightarrow & \mathbf{Div}(U) \\ D = \sum a_i Z_i & \longmapsto & D_{|U} = \sum_{Z_i \cap U \neq \emptyset} a_i (Z_i \cap U) \end{array}$$

To every irreducible divisor is a non-archimedean valuation $v_Z : K(X)^{\times} \to \mathbb{Z}$, which measures the order of cancellation or the pole order of $f \in K(X)^{\times}$ at the generic point of $Z$. The principal divisor associated to a function $f \in K(X)^{\times}$ is $div(f) = \sum_{\text{Z irreducible}} v_Z(f)$.

For each divisor $D$ we have a subsheaf $\mathcal{O}_X(D)$ with sections on each open set $U \subset X$ are

$$\mathcal{O}_X(D) = \{f \in K(X)^{\times} / div(f)_{|U} \succ -D_{|U}\} \cup \{0\}.$$

The bundle $\mathcal{O}_X(D)$ is the sheaf of sections of a line bundle is also noted that $\mathcal{O}_X(D)$. The general theorem of this theory is:

**Theorem 2.3.** *To each smooth variety $X$, it corresponds an exact sequence:*

$$1 \to \mathcal{O}(X)^{\times} \to K(X)^{\times} \xrightarrow{div} \boldsymbol{Div}(X) \xrightarrow{D \mapsto \mathcal{O}_X(D)} \boldsymbol{Pic}(X) \to 1.$$

Let denote $L_1 = \pi^* L \otimes_{\mathcal{O}_{Y\times\mathbb{A}^1}} \mathcal{O}_{Y\times\mathbb{A}^1}(s_0(Y))$. It's a line bundle over $Y \times \mathbb{A}^1$ whose sections are rational sections of $L$ with at worst a simple pole along $s_0(Y)$ and which are regular everywhere else.

**Lemma 2.4.** *There is an isomorphism between the localisation exact sequence and the following one:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{W}^n(Y\times\mathbb{A}^1, L_1) & \xrightarrow{j^*} & \mathcal{W}^n(Y\times\mathbb{G}_m, L_1) & \xrightarrow{\partial} & \mathcal{W}^{n+1}_{s_0(Y)}(Y\times\mathbb{A}^1, L_1) & \longrightarrow & 0 \\ & & p \uparrow \cong & & (\pi^*, \langle x\rangle.\pi^*) \uparrow \cong & & \sigma \uparrow \cong & & \\ 0 & \longrightarrow & \mathcal{W}^n(Y, L) & \xrightarrow{i_2} & \mathcal{W}^n(Y,L) \oplus \mathcal{W}^n(Y,L) & \xrightarrow{p_1} & \mathcal{W}^n(Y, L) & \longrightarrow & 0 \end{array}$$

*where $i_1$ and $p_2$ denote the inclusion of the first factor and the projection on the second one, $s_0$ the null section and finally $x$ is the coordinate on $\mathbb{A}^1$ which vanishes at* 0.

Note that the isomorphism in middle of diagrams of this lemma and the lemma is the same $\pi^* L$ and $L_1$ have the same restrictions to $Y \times \mathbb{G}_m$, but the role of factors of the direct sum in the bottom exact sequence is reversed.

**Lemma 2.5.** *Let $\xi : L \xrightarrow{\cong} L_1$ be an isomorphism of line bundles over a variety $X$. Then*

$$\begin{array}{rcl} \xi_\sharp : \mathcal{W}^n(X, L) & \longrightarrow & \mathcal{W}^n(X, L_1) \\ [\mathcal{E}^{\cdot}, \phi^{\cdot}] & \longmapsto & [\mathcal{E}^{\cdot}, (1_{\mathcal{E}^{\cdot\vee}[n]} \otimes \xi) \circ \phi^{\cdot}] \end{array}$$

*is an isomorphism between derived Witt groups which is compatible with restriction to open subsets and to localisation long exact sequences.*

We identify $\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^1}(0)$. But $\mathbb{P}^1$ is the union of two open subsets $\mathbb{A}^1_0 = \mathbf{Spec}(K[x])$ and $\mathbb{A}^1_\infty = \mathbf{Spec}(K[x^{-1}])$. We have $\mathcal{O}_X(0)(k[x]) = x^{-1}k[x]$ and $\mathcal{O}_X(0)(k[x^{-1}]) = xk[x^{-1}]$.

*Proof of theorem 2.2.* For $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, we identify $\mathcal{O}_{\mathbb{P}^1}(1) \cong t^{-1} \cdot \mathcal{O}_{\mathbb{P}^1} = L(0)$, all germs of rational functions with at worst a simple pole at 0 and regular elsewhere. Then the localisation sequence becomes:

$$0 \longrightarrow \mathcal{W}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow \mathcal{W}^0(\mathbb{G}_m) \xrightarrow{(\beta_0, \beta_\infty)} \mathcal{W}^0_0(\mathbb{P}^1, t^{-1} \cdot \mathcal{O}_{\mathbb{P}^1}) \oplus \mathcal{W}^0_\infty(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{\partial} \mathcal{W}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow 0.$$

Here we have $(\beta_0, \beta_\infty) : \mathcal{W}^0(\mathbb{G}_m) \to \mathcal{W}(k) \oplus \mathcal{W}(k)$, but $\mathcal{W}^0(\mathbb{G}_m) \cong \mathcal{W}(k)\langle 1\rangle \oplus \mathcal{W}(k)\langle t\rangle$. Thus $\beta_0 : a\langle 1\rangle + b\langle t\rangle \longmapsto a$ $\beta_\infty : a\langle 1\rangle + b\langle t\rangle \longmapsto b$. Then $(\beta_0, \beta_\infty)$ is an isomorphism and its kernel is $\mathbf{ker}(\beta_0, \beta_\infty) = \mathcal{W}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$, and its cokernel is $\mathbf{coker}(\beta_0, \beta_\infty) = \mathcal{W}^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 0$. □

# 3 Conclusion

Arason proved that: if $k$ is a field of characteristic not 2 and $n \geq 1$ then $W(\mathbb{P}^n_k) = W(k)$. In 90's Balmer introduced $W^n(X)$, where $X$ is a derived and more general triangulated categories, which have a lot of applications, see for example [6]. Later, Walter proved a projective bundle theorem, which allowed the calculation of $W^i(\mathbb{P}^r_X, \mathcal{O}(m))$ where $X$ is a scheme containing $\frac{1}{2}$, $r \geq 1$, $m \in \mathbb{Z}/2$,

$\mathbb{P}_X^r$ is the $r$-projective space over $X$ and $\mathcal{O}(m) \in Pic(\mathbb{P}_X^r)/2$ [Picard group].

In this paper, we calculate $W^n(\mathbb{P}^1)$ using the famous Balmer's localization sequence, a simple method which permits us to eliminate some hardness. The mentioned method opens the road to find, with real few geometric complexities, $W^n(\mathbb{P}^2)$ and $W^n(\mathbb{P}^3)$. That is our actual objective.

# Competing Interests

The authors declare that no competing interests exist.

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