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A Note on Guaranteed Stable Recovery of Sparse Signal in Compressed Sensing via the RIP of Order s

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Abstract

In this paper, we shall continue a study of the CS-recovery of signals studied in [1]. Under the assumption that a $m \times n$ matrix A obeys the RIP of order s we decompose the space of unknown vectors into sets $M_0,\ M_1, \cdots, M_7$ defined by a bias function p_x on a good location $T_0 = \{1, 2, \cdots, s\}$ and research a good condition of CS-recovery.

Keywords: Compressed sensing; restricted isometry property; sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing(CS). For a signal $\boldsymbol{x} \in \boldsymbol{R}^n$, let $\|\boldsymbol{x}\|_0$ be the l_0-norm of $\boldsymbol{x}$, which is defined to be the number of nonzero coordinates, $\|\boldsymbol{x}\|_1$ be the l_1-norm of $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_2$ be the l_2-norm of $\boldsymbol{x}$. Let $\boldsymbol{x}$ be a sparse or nearly sparse vector. Compressed sensing aims to recover a high-dimensional signal (for example: images signal, voice signal, code signal...etc.) from only a few samples or linear measurements. The efficient recovery of sparse signals has been a very active field in applied mathematics, statistics, machine learning and signal processing. Formally, one considers the following model:

$$\boldsymbol{y} = A\boldsymbol{x} + \boldsymbol{z}, \tag{1.1}$$

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where A is a $m \times n$ matrix($m < n$) and $\boldsymbol{z}$ is an unknown noise term.

Our goal is to reconstruct an unknown signal $\boldsymbol{x}$ based on A and $\boldsymbol{y}$ given. Then we consider reconstructing $\boldsymbol{x}$ as the solution $\boldsymbol{x}^\star$ to the optimization problem

$$\min_{\boldsymbol{x}} \|\boldsymbol{x}\|_1, \quad \text{subject to } \ \|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \varepsilon, \tag{1.2}$$

where ε is an upper bound on the the size of the noisy contribution.
In fact, a crucial issue is to research good conditions under which the inequality

$$\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq C_0\|\boldsymbol{x} - \boldsymbol{x}_T\|_1 + C_1\varepsilon, \tag{1.3}$$

for suitable constants C_0 and C_1, where T is any location of $\{1, 2, \cdots, n\}$ with number $|T| = s$ of elements of T and $\boldsymbol{x}_T$ is the restriction of $\boldsymbol{x}$ to indices in T. One of the most generally known condition for CS theory is the restricted isometry property(RIP) introduced by [2]. When we discuss our proposed results, it is an important notion. The RIP needs that subsets of columns of A for all locations in $\{1, 2, \cdots, n\}$ behave nearly orthonormal system. In detail, a matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1 - \delta)\|\boldsymbol{a}\|_2^2 \leq \|A\boldsymbol{a}\|_2^2 \leq (1 + \delta)\|\boldsymbol{a}\|_2^2 \tag{1.4}$$

for all s-sparse vectors $\boldsymbol{a}$. A vector is said to be an s-sparse vector if it has at most s nonzero entries. The minimum δ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by δ_s.

Many researchers has been shown that the l_1 optimization can recover an unknown signal in noiseless cases and in noisy cases under various sufficient conditions on δ_s or δ_{2s} when A obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [3]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [4]. Others, $\delta_{2s} < 0.4652$ is used in [5], $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large in [6], $\delta_{2s} < 0.4734$ for the case such that s is very large in [5] and $\delta_s < 0.307$ in [7]. In a recent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ [8]. J. Ji and J. Peng have improved the sufficient condition to $\delta_s < 0.308$ [9]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case [10]. T. Cai and A. Zhang have improved the sufficient condition to δ_k in case of $k \geq \frac{4}{3}s$, in particular, $\delta_{2s} < 0.707$ [11]. By using a rescaling method, H. Inoue has obtained the sufficient conditions of $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$ in [12].

Recently, In [1] we have researched good conditions for the recovery of sparse signals by investigating the difference between the l_∞-norm of $\boldsymbol{h} \equiv \boldsymbol{x}^\star - \boldsymbol{x}$ and the mean $\frac{|h_1|+|h_2|+\cdots+|h_s|}{s}$ of $\{|h_1|, \cdots, |h_s|\}$. In more details, we considered a function p on $T_0 \equiv \{1, 2, \cdots, s\}$ defined by

$$p(r) = \frac{|h_1| + |h_2| + \cdots + |h_r|}{|h_1| + |h_2| + \cdots + |h_s|}, \quad r = 1, 2, \cdots, s,$$

where the index of $\boldsymbol{h}$ is sorted by $|h_1| \geq |h_2| \geq \cdots \geq |h_n|$ and have shown that for $c > 1$ and $\frac{c}{s} < p(1)$ if A obeys the RIP of order $\frac{2s}{c}$ and $\delta_{\frac{2s}{c}} < \frac{1}{1+\sqrt{\frac{2-p(r_c)}{p(r_c)}}}$, then we have stable recovery of approximately sparse signals, where r_c is a natural number such that $\frac{c}{s}(r_c - 1) < p(r_c) < \frac{c}{s}r_c$, $2 \leq r_c < \frac{s}{c}$. But, the function p on T_0 and r_c depend on $\boldsymbol{x}$. Furthermore r_c is not easily searched. In this paper, in order to compensate for these defects, we decompose $K_\varepsilon(\boldsymbol{y}, A) \equiv \{\boldsymbol{x} \in \boldsymbol{R}^n;\ \ \|\boldsymbol{y} - A\boldsymbol{x}\|_2 \leq \varepsilon\}$ into the following subsets $\{M_0, M_1, \cdots, M_7\}$:

$$
\begin{aligned}
M_0 &= \left\{ \boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{5}s\right) \le \frac{2}{5} \right\},\\
M_1 &= \left\{ \boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{5}s\right) > \frac{2}{5} \text{ and } p_{\boldsymbol{x}}\left(\frac{1}{4}s\right) \le \frac{1}{2} \right\},\\
&\vdots\\
M_k &= \left\{ \boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{k+3}{20}s\right) > \frac{k+3}{10} \text{ and } p_{\boldsymbol{x}}\left(\frac{k+4}{20}s\right) \le \frac{k+4}{10} \right\},\quad 2 \le k \le 6,\\
M_7 &= \left\{ \boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{2}s\right) = 1 \right\}
\end{aligned}
$$

by deviding $T_0 = \{1, 2, \cdots, s\}$ into $T_0 \cap [1, \frac{s}{5})$, $T_0 \cap (\frac{k+3}{20}, \frac{k+4}{20}](k = 1, \cdots, 6)$ and $T_0 \cap (\frac{1}{2}s, s]$, and we show for any $\boldsymbol{x} \in M_k(k = 1, 2, \cdots, 7)$ that if A obeys the RIP of order s and $\delta_s < \frac{1}{1+\sqrt{\frac{20}{k+3}-1}}$, then the inequality (1.3) holds. We also state in Section 2 the existence of CS-solution.

2 CS-Solution

In this section, we discuss the existence of CS-solutions mathematically.

Let a $m \times n$ matrix A $(m < n)$ and a data $\boldsymbol{y} \in \boldsymbol{R}^m$ be given. We define closed convex subsets of $\boldsymbol{R}^n$ by

$$
\begin{aligned}
K_0(\boldsymbol{y}, A) &= \{\boldsymbol{x} \in \boldsymbol{R}^n;\ \ \boldsymbol{y} = A\boldsymbol{x}\},\\
K_\varepsilon(\boldsymbol{y}, A) &= \{\boldsymbol{x} \in \boldsymbol{R}^n;\ \ \|\boldsymbol{y} - A\boldsymbol{x}\|_2 \le \varepsilon\},\quad \varepsilon > 0.
\end{aligned}
$$

When $K_0(\boldsymbol{y}, A) \ne 0$, that is, $\boldsymbol{y} \in A\boldsymbol{R}^n$, then $K_0(\boldsymbol{y}, A)$ and $K_\varepsilon(\boldsymbol{y}, A)$ are

$$K_0(\boldsymbol{y}, A) = \boldsymbol{x}_0 + \ker A$$

for some vector $\boldsymbol{x}_0 \in K_0(\boldsymbol{y}, A)$, where $\ker A \equiv \{\boldsymbol{x} \in \boldsymbol{R}^n;\ \ A\boldsymbol{x} = \boldsymbol{0}\}$. For example, if the rank $r(A)$ of A equals m, then AA^* is invertible and $A\left(A^*\left(AA^*\right)^{-1}\boldsymbol{y}\right) = \boldsymbol{y}$. Hence, $A^*\left(AA^*\right)^{-1}\boldsymbol{y} \in K_0(\boldsymbol{y}, A)$. Let $\boldsymbol{y} \notin A\boldsymbol{R}^n$. Since $A\boldsymbol{R}^n$ is a closed subspace of $\boldsymbol{R}^n$, there exists a unique vector $\boldsymbol{y}_0 \in A\boldsymbol{R}^n$ such that $\|\boldsymbol{y} - \boldsymbol{y}_0\|_2 = \min\{\|\boldsymbol{y} - A\boldsymbol{x}\|_2;\ \ \boldsymbol{x} \in \boldsymbol{R}^n\}$. Then $\boldsymbol{y}_0$ is a vector in $A\boldsymbol{R}^n$ such that $\boldsymbol{y} - \boldsymbol{y}_0$ is a vector in the orthogonal complement $(A\boldsymbol{R}^n)^\perp$ of $A\boldsymbol{R}^n$. It is clear that $K_\varepsilon(\boldsymbol{y}, A) \ne \emptyset$ if and only if $\|\boldsymbol{y} - \boldsymbol{y}_0\|_2 \le \varepsilon$. In this paper, we assume that $K_0(\boldsymbol{y}, A) \ne \emptyset$ in noiseless cases and $K_\varepsilon(\boldsymbol{y}, A) \ne \emptyset$ in noise cases. We show the existence of CS-solutions.

For any $t > 0$ we put

$$D_t = \{\boldsymbol{x} \in \boldsymbol{R}^n;\ \ \|\boldsymbol{x}\|_1 \le t\}.$$

Then AD_t is a closed convex subset of $A\boldsymbol{R}^n$ such that $A\left(\partial D_t\right) = \partial AD_t$, where ∂K is a boundary of a set K. Assume that $\boldsymbol{y}_0 \notin AD_t$. Then there exists a vector $\boldsymbol{x}_t$ in ∂D_t such that $\|\boldsymbol{y} - A\boldsymbol{x}_t\|_2 = \min\{\|\boldsymbol{y}_0 - A\boldsymbol{x}\|_2;\ \ \boldsymbol{x} \in D_t\}$. Since

$$\|\boldsymbol{y} - A\boldsymbol{x}_t\|_2^2 = \|\boldsymbol{y} - \boldsymbol{y}_0\|_2^2 + \|\boldsymbol{y}_0 - A\boldsymbol{x}_t\|_2^2,$$

we have

$$\|\boldsymbol{y} - A\boldsymbol{x}_t\|_2 = \min\{\|\boldsymbol{y} - A\boldsymbol{x}\|_2;\ \ \boldsymbol{x} \in D_t\},$$

which implies that there exists a vector $\boldsymbol{x}_t^\star$in $(\boldsymbol{x}_t + \ker A) \cap D_t$ such that

$$\|\boldsymbol{x}_t^\star\|_1 \leq \|\boldsymbol{x}_t + \boldsymbol{x}\|_1, \quad {}^\forall \boldsymbol{x} \in \ker A.$$

Thus we have the following:

Proposition 2.1. Suppose that $K_\varepsilon(\boldsymbol{y}, A) \neq \emptyset$. Then there exists a positive number t_0 such that

$$\|\boldsymbol{y}_0 - A\boldsymbol{x}_{t_0}\|_2^2 = \varepsilon^2 - \|\boldsymbol{y} - \boldsymbol{y}_0\|_2^2$$

and the vector $\boldsymbol{x}_{t_0}^\star$ determined by $\boldsymbol{x}_{t_0}$ equals the CS-solution $\boldsymbol{x}^\star$. In particular, in noiseless cases, $\boldsymbol{x}^\star = \boldsymbol{x}_{t_0}^\star$, where t_0 is a positive number satisfying $\boldsymbol{y}_0 = A\boldsymbol{x}_{t_0}$.

3 Recovery of CS

Take an arbitrary $\boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A)$. We denote by $\boldsymbol{x}^T$ a vector obtained by changing coefficients of $\boldsymbol{x}$ as follows;

$$|h_1| \geq |h_2| \geq \cdots \geq |h_n|,$$

where $\boldsymbol{h} = (h_1, h_2, \cdots h_n) \equiv \boldsymbol{x}^\star - \boldsymbol{x}^T$. Let $T_0 = \{1, 2, \cdots, s\}$ and we define a function $p_{\boldsymbol{x}}(r)$ on T_0 depending on $\boldsymbol{x}$ by

$$p_{\boldsymbol{x}}(r) = \frac{|h_1| + |h_2| + \cdots + |h_r|}{\|\boldsymbol{h}_{T_0}\|_1}, \quad r \in T_0.$$

By deviding $T_0 = \{1, 2, \cdots, s\}$ into $T_0 \cap [1, \frac{s}{5}]$, $T_0 \cap (\frac{k+3}{20}s,\ \frac{k+4}{20}s]$ $(k = 1, \cdots, 6)$ and $T_0 \cap (\frac{1}{2}s, s]$, we decomposed $K_\varepsilon(\boldsymbol{y}, A)$ into the following subsets $\{M_0, M_1, \cdots, M_7\}$;

$$\begin{aligned}
M_0 &= \left\{\boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{5}s\right) \leq \frac{2}{5}\right\}, \\
M_1 &= \left\{\boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{5}s\right) > \frac{2}{5} \text{ and } p_{\boldsymbol{x}}\left(\frac{1}{4}s\right) \leq \frac{1}{2}\right\}, \\
&\vdots \\
M_k &= \left\{\boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{k+3}{20}s\right) > \frac{k+3}{10} \text{ and } p_{\boldsymbol{x}}\left(\frac{k+4}{20}s\right) \leq \frac{k+4}{10}\right\},\ \ 2 \leq k \leq 6, \\
M_7 &= \left\{\boldsymbol{x} \in K_\varepsilon(\boldsymbol{y}, A);\quad p_{\boldsymbol{x}}\left(\frac{1}{2}s\right) = 1\right\}.
\end{aligned}$$

Then, $K_\varepsilon(\boldsymbol{y}, A) = \bigcup_{k=0}^7 M_k$ and $M_i \cap M_j = \emptyset (i \neq j)$. (Figure 1)

Using the function $p_{\boldsymbol{x}}(r)$ on T_0, we obtain a similar result to that of ([1] Theorem 2.1):

Theorem 3.1. Take an arbitrary $\boldsymbol{x} \in M_k$ $(k = 1, 2, \cdots 7)$. Assume that A obeys the RIP of order s and $\delta_s < \frac{1}{1+\sqrt{\frac{20}{k+3}-1}}$. Then,

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \leq C_0^{(k)}\|\boldsymbol{x} - \boldsymbol{x}_s\|_1 + C_1^{(k)}\varepsilon,$$

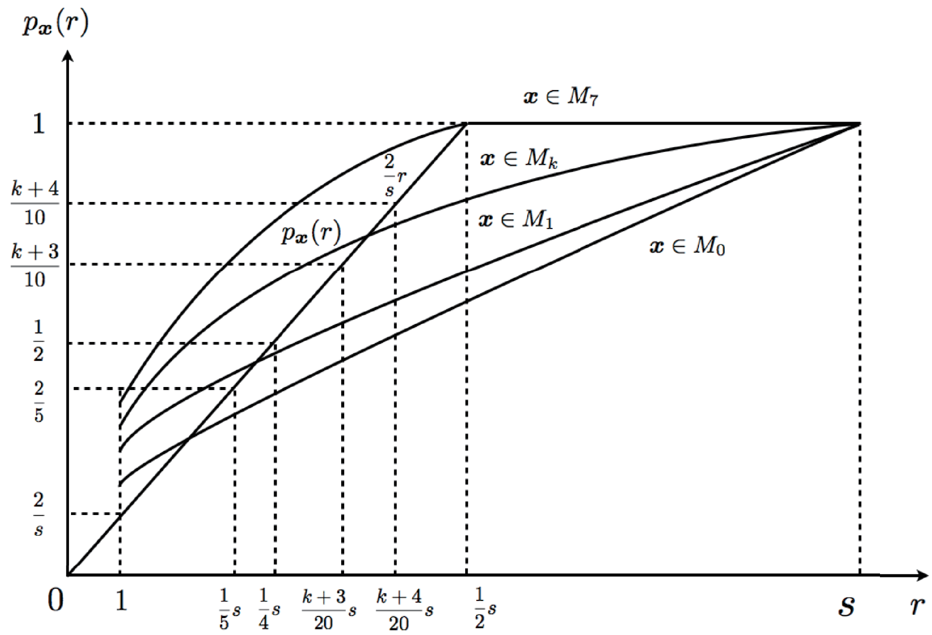

Figure 1: $\{M_0, M_1, \cdots, M_7\}$

where $\boldsymbol{x}_s$ is a vector consisting of the s-large entries of $\boldsymbol{x}$ in magnitude and

$$C_0^{(k)} = \frac{4\sqrt{\frac{20}{k+3} - 1} \cdot \delta_s}{1 - \left(1 + \sqrt{\frac{20}{k+3} - 1}\right)\delta_s},$$

$$C_1^{(k)} = \frac{2\sqrt{1+\delta_s}\sqrt{s}}{\sqrt{\frac{k+3}{20}}\left(1 - \left(1 + \sqrt{\frac{20}{k+3} - 1}\right)\delta_s\right)}.$$

Proof. Take an arbitrary $\boldsymbol{x} \in M_k$. Let r_k be a natural number such that

$$\frac{k+3}{20}s < r_k \leq \frac{k+4}{20}s \quad \text{and}$$
$$\frac{2}{s}(r_k - 1) < p_{\boldsymbol{x}}(r_k) \leq \frac{2}{s}r_k. \tag{3.1}$$

Then,

$$\frac{k+3}{10} < p_{\boldsymbol{x}}(r_k) \leq \frac{k+4}{10}. \tag{3.2}$$

We put

$$\alpha = \frac{\|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1}{s}.$$

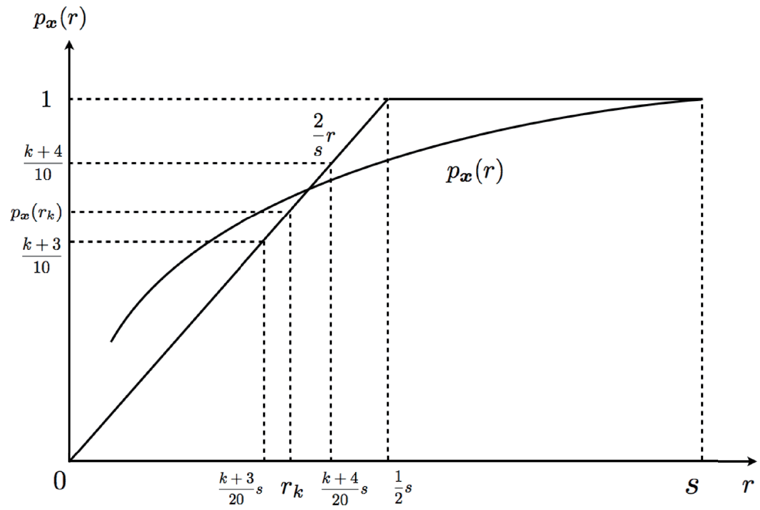

Figure 2: $\boldsymbol{x} \in M_k$

Let $T_1 = \{1, 2, \cdots, r_2\}$ and $T_2 = \{r_2 + 1, \cdots, n\}$ be a decomposition of $\{1, 2, \cdots, n\}$. By (3.1) and (3.2) we have

$$\|\boldsymbol{h}_{T_2}\|_\infty \leq \frac{p_x(r_2)}{r_2}\|\boldsymbol{h}_{T_0}\|_1 \leq 2\alpha. \tag{3.3}$$

By the definition of CS optimization (1.2), we have

$$\|\boldsymbol{h}_{T_0^c}\|_1 \leq \|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1. \tag{3.4}$$

Hence it follows from (3.3) and (3.4) that

$$\begin{aligned}\|\boldsymbol{h}_{T_2}\|_1 &= \|\boldsymbol{h}_{T_0^c}\|_1 + \|\boldsymbol{h}_{T_0\cap T_2}\|_1 \\ &\leq \alpha s + (1 - p_{\boldsymbol{x}}(r_k))\,\|\boldsymbol{h}_{T_0}\|_1 \\ &\leq (2 - p_{\boldsymbol{x}}(r_k))\,\alpha s \\ &\leq 2\alpha\left(1 - \frac{k+3}{20}\right)s,\end{aligned}$$

which implies by [1] Lemma 1.1 and the Cai idea [4] that there exist $\{\lambda_i\}_{1\leq i\leq N}$ and $\{\boldsymbol{u}_i\}_{1\leq i\leq N}$ such that

$$\boldsymbol{h}_{T_2} = \sum_{i=1}^{N}\lambda_i\boldsymbol{u}_i,$$

where

$$\begin{aligned}
&0 \le \lambda_i \le 1, \quad \sum_{i=1}^{N} \lambda_i = 1, \\
&\mathrm{supp}\, \boldsymbol{u}_i \subset T_2, \quad |\mathrm{supp}\, \boldsymbol{u}_i| \le \left(1 - \frac{k+3}{20}\right) s \\
&\|\boldsymbol{u}_i\|_\infty \le 2\alpha. \qquad (3.5)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\|\boldsymbol{u}_i\|_2 &\le \|\boldsymbol{u}_i\|_\infty \sqrt{|\mathrm{supp}\, \boldsymbol{u}_i|} \\
&\le 2\alpha\sqrt{s}\sqrt{1 - \frac{k+3}{20}}, \\
|T_1| + |\mathrm{supp}\, \boldsymbol{u}_i| &\le r_k + \left(1 - \frac{k+3}{20}\right) s \le s
\end{aligned}$$

and

$$\begin{aligned}
\alpha s &= \|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1 \\
&= \frac{1}{p_{\boldsymbol{x}}(r_k)} \|\boldsymbol{h}_{T_1}\|_1 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1 \\
&\le \frac{\sqrt{r_k}}{p_{\boldsymbol{x}}(r_k)} \|\boldsymbol{h}_{T_1}\|_2 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1 \\
&\le \frac{\sqrt{s}}{2\sqrt{\frac{k+3}{20}}} \|\boldsymbol{h}_{T_1}\|_2 + 2\|\boldsymbol{x} - \boldsymbol{x}_s\|_1,
\end{aligned}$$

which implies since A obeys the RIP of order s that

$$\begin{aligned}
(1-\delta_s)\|\boldsymbol{h}_{T_1}\|_2^2 &\le \|A\boldsymbol{h}_{T_1}\|_2^2 \\
&\le |\langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}\rangle| + |\langle A\boldsymbol{h}_{T_1}, A\boldsymbol{h}_{T_2}\rangle| \\
&\le \sqrt{1+\delta_s}\|\boldsymbol{h}_{T_1}\|_2 \cdot 2\varepsilon + \sum_{i=1}^{N} \lambda_i \, |\langle A\boldsymbol{h}_{T_1}, A\boldsymbol{u}_i\rangle| \\
&\le 2\sqrt{1+\delta_s}\varepsilon\|\boldsymbol{h}_{T_1}\|_2 + \sum_{i=1}^{N} \lambda_i \delta_s \|\boldsymbol{h}_{T_1}\|_2 \|\boldsymbol{u}_i\|_2 \\
&\le 2\sqrt{1+\delta_s}\varepsilon\|\boldsymbol{h}_{T_1}\|_2 \\
&\quad + \delta_s \|\boldsymbol{h}_{T_1}\|_2 \left(\frac{1}{2\sqrt{\frac{k+3}{20}}} \|\boldsymbol{h}_{T_1}\|_2 + \frac{2}{\sqrt{s}} \|\boldsymbol{x} - \boldsymbol{x}_s\|_1 \right) 2\sqrt{1 - \frac{k+3}{20}} \\
&= 2\sqrt{1+\delta_s}\varepsilon\|\boldsymbol{h}_{T_1}\|_2 + \delta_s \sqrt{\frac{20}{k+3} - 1}\|\boldsymbol{h}_{T_1}\|_2^2 \\
&\quad + \frac{4\delta_s}{\sqrt{s}} \sqrt{1 - \frac{k+3}{20}} \|\boldsymbol{x} - \boldsymbol{x}_s\|_1 \|\boldsymbol{h}_{T_1}\|_2.
\end{aligned}$$

Since

$$\left(1 + \sqrt{\frac{20}{k+3} - 1}\right) \delta_s < 1,$$

we have

$$\|\boldsymbol{h}_{T_1}\|_2 \le \frac{2\sqrt{1+\delta_s}\varepsilon + \frac{4\delta_s}{\sqrt{s}}\sqrt{1-\frac{k+3}{20}}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1}{1-\left(1+\sqrt{\frac{20}{k+3}-1}\right)\delta_s},$$

which implies that

$$\begin{aligned}
\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 &\le \|\boldsymbol{x}-\boldsymbol{x}^\star\|_1 \\
&= \|\boldsymbol{h}_{T_0}\|_1 + \|\boldsymbol{h}_{T_0^c}\|_1 \\
&\le 2\|\boldsymbol{h}_{T_0}\|_1 + 2\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 \\
&\le \frac{2\sqrt{r_k}}{p_{\boldsymbol{x}}(r_k)}\|\boldsymbol{h}_{T_1}\|_2 + 2\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 \\
&\le \frac{\sqrt{s}}{\sqrt{\frac{k+3}{20}}}\left(\frac{2\sqrt{1+\delta_s}\varepsilon + \frac{4}{\sqrt{s}}\sqrt{1-\frac{k+3}{20}}\delta_s\|\boldsymbol{x}-\boldsymbol{x}_s\|_1}{1-\left(1+\sqrt{\frac{20}{k+3}-1}\right)\delta_s}\right) \\
&\quad +2\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 \\
&= \frac{2\sqrt{1+\delta_s}\sqrt{s}}{\sqrt{\frac{k+3}{20}}\left(1-\left(1+\sqrt{\frac{20}{k+3}-1}\right)\delta_s\right)}\varepsilon \\
&\quad + \frac{4\sqrt{\frac{20}{k+3}-1}\cdot\delta_s}{1-\left(1+\sqrt{\frac{20}{k+3}-1}\right)\delta_s}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1.
\end{aligned}$$

This completes the proof.

We state concretely the following case:

(i) Take an arbitrary $\boldsymbol{x} \in M_1$. If $\delta_s < \frac{1}{3}$, then

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le \frac{8\delta_s}{1-3\delta_s}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 + \frac{2\sqrt{5}\sqrt{1+\delta_s}\sqrt{s}}{1-3\delta_s}\varepsilon.$$

(ii) Take an arbitrary $\boldsymbol{x} \in M_2$. If $\delta_s < \frac{\sqrt{3}-1}{2} \approx 0.366$, then

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le \frac{4\sqrt{3}\delta_s}{1-(1+\sqrt{3})\delta_s}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 + \frac{4\sqrt{1+\delta_s}\sqrt{s}}{1-(1+\sqrt{3})\delta_s}\varepsilon.$$

(iii) Take an arbitrary $\boldsymbol{x} \in M_7$. If $\delta_s < \frac{1}{2}$, then

$$\|\boldsymbol{x}^\star - \boldsymbol{x}\|_2 \le \frac{4\delta_s}{1-2\delta_s}\|\boldsymbol{x}-\boldsymbol{x}_s\|_1 + \frac{2\sqrt{2}\sqrt{1+\delta_s}\sqrt{s}}{1-2\delta_s}\varepsilon.$$

Though we have decomposed $K_\varepsilon(\boldsymbol{y}, A)$ into $M_k (k = 0, 1, \cdots, 7)$ in this paper, we may consider the other decompositions of $K_\varepsilon(\boldsymbol{y}, A)$.

4 Conclusion

In a previous paper [1], we have discussed sufficient conditions of isometry constant δ by investigating a bias function p_x defined by each unknown vector $\boldsymbol{x}$. In this paper, we decompose the space of unknown vectors into sets $M_0,\ M_1, \cdots, M_7$ defined by the bias function p_x. More precisely, when

$\boldsymbol{x}$ is contained in M_k $(1 \leq k \leq n)$, the sufficient condition of δ_s is improved, and so this method is useful. When $\boldsymbol{x} \in M_0$, the sufficient condition of δ_s is not improved by this method. We think that this method is more usable than a previous one in [1].

Competing Interests

The author declares that no competing interests exist.

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