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A Voronovskaya Type Theorem for Bernstein-Durrmeyer Type Operators

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Abstract

Bernstein operators constitute a powerful tool allowing one to replace many inconvenient calculations performed for continuous functions by more friendly calculations on approximating polynomials. In this note we study a modification of Bernstein type operators and prove in particular that they satisfy Voronovskaya type theorems.

Keywords: Bernstein-Durrmeyer operators; voronoskaya type theorem.

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1 Introduction

Approximation issues appear in many branches of mathematics and they find a huge number of applications beyond mathematics. They originate in the celebrated Weierstrass Theorem to the effect that any continuous function on a compact interval can be uniformly approximated by polynomials.

Weierstrass Theorem is non-constructive in the sense that it provides no algorithm for construction

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of approximating polynomials. This problem has been taken on by Bernstein [1] who constructed in 1912 for a given continuous function $f : [0, 1] \to \mathbb{R}$ a sequence of polynomials $B_n(f, x)$ converging uniformly to $f(x)$

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}.$$

These polynomials are nowadays called *Bernstein polynomials*.

Slightly more generally, B_n's as functions of argument f can be viewed as linear operators from the space of continuous functions on $[a, b]$ to the space of polynomials in one variable of degree at most $n \in \mathbb{N}$.

In the course of time Bernstein operators have been modified and generalized, see eg. [2] and references therein. In particular one studies Bernstein-Durrmeyer type operators defines as follows, [3].

For the rest of the paper we fix a positive real number ρ.

Definition 1.1. Let n be a positive integer and f be a Lebesgue integrable function on $[0, 1]$. Define an operator U_n^ρ for $x \in [0, 1]$ by

$$U_n^\rho(f,x) = \sum_{k=0}^{n} F_{n,k}^\rho(f) \cdot p_{n,k}(x), \text{ where}$$

$$F_{n,k}^\rho(f) = \begin{cases} \text{f}(0) & \text{for } k = 0, \\ \int_0^1 f(t) \cdot \mu_{n,k}^\rho(t)dt & \text{for } k = 1, 2, 3, ..., n-1, \\ \text{f}(1) & \text{for } k = n, \end{cases}$$

$$\mu_{n,k}^\rho(t) = \frac{t^{k\cdot\rho-1}(1-t)^{(n-k)\cdot\rho-1}}{B(k\cdot\rho, (n-k)\cdot\rho)},$$

and B is Euler beta function.

These operators are defined on the space of Lebesgue integrable functions which is considerably bigger than the space of continuous functions. Nevertheless these operators enjoy similar properties as Bernstein operators.

2 Main Results

We begin by showing that $U_n^\rho(f)$ are approximation polynomials for continuous functions. This fact is surely well–known, but we were not able to find it in the literature and include the proof for sake of completeness.

Note that the convergence of Durrmeyer operators in multivariate setings and with respect to various measures has been extensively investigated by Berdysheva in [4].

Theorem 2.1. *For any f continuous function on $[0, 1]$ the sequence $U_n^\rho(f)$ is converging uniformly to f.*

Proof. We will use the famous Korovkin theorem. To this end we need to have uniformly convergence $U_n^\rho(e_i, x) \to e_i$ for $i \in \{0, 1, 2\}$, where $e_i(x) = x^i$. By definition for all n we obtain $U_n^\rho(e_0, x) = e_0(x)$ and $U_n^\rho(e_1, x) = e_1(x)$. This implies in particular that $U_n^\rho(e_i)$ uniformly converges to e_i on $[0, 1]$ for $i \in \{0, 1\}$. We need to show the uniform convergence for $i = 2$. We have pointwise convergence

$$\lim_{n\to\infty} U_n^\rho(e_2, x) = \lim_{n\to\infty} \frac{\rho(n-1)x^2 + (\rho+1)x}{n\rho + 1} = e_2(x).$$

Let

$$M_n = \sup_{0\le x\le 1} |U_n^\rho(e_2, x) - e_2(x)|.$$

Then

$$M_n = \sup_{0\le x\le 1} \left| \frac{\rho(n-1)x^2 + (\rho+1)x}{n\rho + 1} - x^2 \right| = \sup_{0\le x\le 1} \frac{\rho+1}{n\rho+1} \left|-x^2 + x\right| = \frac{1}{4} \cdot \frac{\rho+1}{n\rho+1}.$$

It follows that

$$\lim_{n\to\infty} M_n = 0.$$

That means that the convergence

$$U_n^\rho(e_2, x) \longrightarrow e_2(x)$$

is uniform. From the Korovkin theorem, when $U_n^\rho(e_i) \to e_i$ converges uniformly on $[0, 1]$ for $i \in \{0, 1, 2\}$, then $U_n^\rho(f) \to f$ converges uniformly for every continuous function f. That ends the proof. □

The second result is the verification that Voronovskaya type theorem holds for Bernstein-Durrmeyer operators. In 1932 Voronovskaya obtained a result dealing with the asymptotic behavior of the classical Bernstein operators. Her theorem is stated in the book of De Vore and Lorentz [5] as follows.

Theorem 2.2. *If f is bounded on $[0, 1]$, differentiable in some neighborhood of x_0 and has second derivative $f''(x_0)$ for some $x_0 \in [0, 1]$, then*

$$\lim_{n\to\infty} n[B_n(f, x_0) - f(x_0)] = \frac{x_0(1-x_0)}{2} \cdot f''(x_0).$$

If $f \in C^2[0, 1]$, then the convergence is uniform.

This result attracted a lot of attention a few decades later and many authors inspired by Voronovskaya's theorem proved similar asymptotic properties for other groups of operators. All these theorems are jointly called *Voronovskaya type theorems.*

A result of this type has been in particular claimed by Păltănea in [6]. In the paper of Păltănea the proof is missing and the theorem is stated with the right hand side coefficient $\frac{1}{2}$ instead $\frac{\rho+1}{2\rho}$. Therefor we present the correct formulation and the complete proof below.

Theorem 2.3. *If f is a continuous function on $[0, 1]$ and twice differentiable in every $x \in (0, 1)$, then we have*

$$\lim_{n\to\infty} n[U_n^\rho(f, x) - f(x)] = \frac{(\rho+1)f''(x)}{2\rho} \cdot x(1-x).$$

Proof. Let f be a continuous function on $[0, 1]$ and twice differrentiable in $(0, 1)$. Using Taylor's formula for fixed $x_0 \in [0, 1]$ we obtain □

$$f(t) = f(x_0) + f'(x_0)(t - x_0) + \frac{1}{2} \cdot f''(x_0)(t - x_0)^2 + (t - x_0)^2 \cdot \varepsilon_{x_0}(t)$$

where $\lim_{t \to x_0} \varepsilon_{x_0}(t) = 0$ and function $t \to \varepsilon_{x_0}(t)$ is continuous on $[0, 1]$. From the definition of U_n^ρ we have

$$U_n^\rho(f, x_0) = f(x_0) \cdot U_n^\rho(e_0, x_0) + f'(x_0) \cdot U_n^\rho(e_1 - x_0, x_0) + \frac{1}{2} \cdot f''(x_0) \cdot U_n^\rho(e_2 - 2x_0 e_1 + x_0^2, x_0) +$$

$$+ U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0) = f(x_0) + f'(x_0) \cdot [U_n^\rho(e_1, x_0) - x_0 \cdot U_n^\rho(e_0, x_0)] +$$

$$+ \frac{1}{2} \cdot f''(x_0) \cdot \left[U_n^\rho(e_2, x_0) - 2x_0 \cdot U_n^\rho(e_1, x_0) + x_0^2 \cdot U_n^\rho(e_0, x_0)\right] + U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0) =$$

$$= f(x_0) + \frac{1}{2} \cdot f''(x_0) \cdot \frac{-(\rho + 1)x_0^2 + (\rho + 1)x_0}{n\rho + 1} + U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0).$$

It follows that

$$n\, [U_n^\rho(f, x_0) - f(x_0)] = n \left[\frac{1}{2} \cdot f''(x_0) \cdot \frac{-(\rho + 1)x_0^2 + (\rho + 1)x_0}{n\rho + 1} + U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0)\right]$$

$$= \frac{1}{2} \cdot f''(x_0) \cdot n \cdot \frac{-(\rho + 1)x_0^2 + (\rho + 1)x_0}{n\rho + 1} + n \cdot U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}.$$

Let

$$A_1^{(n)} = \frac{1}{2} \cdot f''(x_0) \cdot n \cdot \frac{-(\rho + 1)x_0^2 + (\rho + 1)x_0}{n\rho + 1},$$

$$A_2^{(n)} = U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}.$$

Then

$$\lim_{n\to\infty} n[U_n^\rho(f, x_0) - f(x_0)] = \lim_{n\to\infty} A_1^{(n)} + \lim_{n\to\infty} A_2^{(n)}.$$

We compute the first limit limes on the right hand side

$$\lim_{n\to\infty} A_1^{(n)} = \lim_{n\to\infty} \tfrac{1}{2} \cdot f''(x_0) \cdot n \cdot \tfrac{-(\rho+1)x_0^2+(\rho+1)x_0}{n\rho+1}$$

$$= \tfrac{1}{2} \cdot f''(x_0) \cdot \lim_{n\to\infty} n \cdot \tfrac{-(\rho+1)x_0^2+(\rho+1)x_0}{n(\rho+\frac{1}{n})} = \tfrac{1}{2} \cdot f''(x_0) \cdot \lim_{n\to\infty} \tfrac{-(\rho+1)x_0^2+(\rho+1)x_0}{\rho+\frac{1}{n}}$$

$$= \tfrac{1}{2} \cdot f''(x_0) \cdot \lim_{n\to\infty} \tfrac{-(\rho+1)x_0^2+(\rho+1)x_0}{\rho} = \tfrac{(\rho+1)\cdot f''(x_0)}{2\rho} \cdot x_0(1 - x_0).$$

Now, we need to calculate $\lim_{n\to\infty} A_2^{(n)}$. From Cauchy–Schwarz inequality for positive operators we have that

$$|U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0)| \le \sqrt{U_n^\rho((e_1 - x_0)^4, x_0)} \cdot \sqrt{U_n^\rho((\varepsilon_{x_0})^2, x_0)}.$$

It follows that

$$n \cdot |U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0)| \le \sqrt{n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0)} \cdot \sqrt{U_n^\rho((\varepsilon_{x_0})^2, x_0)}.$$

If $\lim_{t\to x_0} \varepsilon_{x_0}(t) = 0$ and $t \to \varepsilon_{x_0}(t)$ is a continuous function, then $t \to (\varepsilon_{x_0}(t))^2$ is a continuous function, too and $\lim_{t\to x_0}(\varepsilon_{x_0}(t))^2 = 0$. Since also $U_n^\rho(f) \to f$ converges uniformly we obtain

$$\lim_{n\to\infty} U_n^\rho((\varepsilon_{x_0})^2, x_0) = 0,$$

so that finally

$$\lim_{n\to\infty} \sqrt{U_n^\rho((\varepsilon_{x_0})^2, x_0)} = 0.$$

Now we take care of the term $\sqrt{n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0)}$. After straightforward computations we have

$$n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0) = n^2 \cdot U_n^\rho((e_4 - 4e_3x_0 + 6e_2x_0^2 - 4e_1x_0^3 + x_0^4, x_0), x_0)$$

$$= \frac{[n^3(3\rho^3+6\rho^2+3\rho)-n^2(6\rho^3+24\rho^2+36\rho+18)]x_0^4}{(n\rho+1)(n\rho+2)(n\rho+3)} + \frac{[-n^3(6\rho^3+12\rho^2+6\rho)+n^2(36\rho^3+72\rho^2+72\rho+36)]x_0^3}{(n\rho+1)(n\rho+2)(n\rho+3)}$$

$$+ \frac{[n^3(3\rho^3+6\rho^2-3\rho)-n^2(7\rho^3+30\rho^2+47\rho+24)]x_0^2}{(n\rho+1)(n\rho+2)(n\rho+3)} + \frac{n^2(\rho+1)(\rho+2)(\rho+3)x_0}{(n\rho+1)(n\rho+2)(n\rho+3)}.$$

The sequence

$$n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0)$$

has a finite limit for all $x_0 \in [0, 1]$. Therefore the limit of the product below exists and we have

$$\lim_{n\to\infty} \left[\sqrt{n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0)} \cdot \sqrt{U_n^\rho((\varepsilon_{x_0})^2, x_0)}\right] = 0.$$

Therefore we obtain

$$0 \le n \cdot |U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0)| \le \sqrt{n^2 \cdot U_n^\rho((e_1 - x_0)^4, x_0)} \cdot \sqrt{U_n^\rho((\varepsilon_{x_0})^2, x_0)}$$

and it follows that

$$\lim_{n\to\infty} \left[n \cdot U_n^\rho((e_1 - x_0)^2 \cdot \varepsilon_{x_0}, x_0)\right] = \lim_{n\to\infty} A_2^{(n)} = 0.$$

So that finally we obtain

$$\lim_{n\to\infty} n[U_n^\rho(f, x_0) - f(x_0)] = \frac{(\rho+1)f''(x_0)}{2\rho} \cdot x_0(1 - x_0).$$

Since this works for arbitrary $x_0 \in (0, 1)$, the proof is finished.

3 Conclusion

We prove that $U_n^\rho(f)$ are approximation polynomials for continuous functions. We also verify that Voronovskaya type theorem holds for Bernstein-Durrmeyer operators. The result stated in the present note is valid for continuous functions only. It seems possible to extend it in an appropriate way to Lebesgue integrable functions. We hope to come back to this problem in the near future.

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Competing Interests

The author declares that no competing interests exist.

References

[1] Bernstein S. Démonstration du théorème de Weierstrass. fondée sur le calcul des probalités. Commun. Soc. Math. Kharkow. 1912;2(13).

[2] Gonska H. Păltănea R. Simultaneous approximation by a class of Bernstein-Durrmeyer operators preserving linear functions. Czechoslovak Math. J. 2010;60(135):783-799.

[3] Durrmeyer JL. Une formule d'inversion de la transformè de Laplace: Applications à la théorie des moments, Thèse de 3e cycle. Faculté des Sciences de l'Université de Paris; 1967.

[4] Berdysheva E. Uniform convergence of Bernstein- Durrmeyer operators with respect to arbitrary measure. J. Math. Anal. Appl. 2012;394:324-336.

[5] Devore RA, Lorentz GG. Constructive approximation. Springer-Verlag, Berlin; 1993.

[6] Gonska H. Păltănea R. A class of Durrmeyer type operators preserving linear functions. Annals of Tiberiu Popoviciu Seminar of Functional Equations. Czechoslovak Math. J. 2007;5:109-118.

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