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Solitons and Periodic Wave Solutions of The (3+1)-dimensional Potential Yu–Toda–Sasa–Fukuyama Equation

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Authors' contributions

This work was carried out in collaboration between the authors. All authors have a good contribution to design the study, and to perform the analysis of this research work. All authors read and approved the final manuscript.

Research Article

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ABSTRACT

In this work we explore an enhanced (G'/G)-expansion method to study the nonlinear evolution equations (NLEEs). Here we derive solitons, singular solitons and periodic wave solutions for the nonlinear (3+1)-dimensional Potential Yu–Toda–Sasa–Fukuyama (YTSF) equation. The obtained results show that the applied equation reveal richness of explicit solitons and periodic solutions. It is shown that the proposed method is effective and can be used for many other NLEEs in mathematical physics.

Keywords: Enhanced (G'/G)-expansion method; YTSF equation; solitons; NLEEs.

Mathematics Subject Classification: 35K99, 35P05, 35P99.

1. INTRODUCTION

NLEEs are encountered in various fields of mathematics, physics, chemistry, biology, engineering and numerous applications. Exact solutions of NLEEs play an important role in

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the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Exact solutions of nonlinear equations graphically demonstrate and allow unscrambling the mechanisms of many complex nonlinear phenomena such as spatial localization of transfer processes, multiplicity or absence steady states under various conditions, existence of peaking regimes and many others. Even those special exact solutions that do not have a clear physical meaning can be used as test problems to verify the consistency and estimate errors of various numerical, asymptotic, and approximate analytical methods. Exact solutions can serve as a basis for perfecting and testing computer algebra software packages for solving NLEEs. It is significant that many equations of physics, chemistry, and biology contain empirical parameters or empirical functions. Exact solutions allow researchers to design and run experiments, by creating appropriate natural conditions, to determine these parameters or functions. Therefore, investigation exact traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the Hirota's bilinear transformation method [1,2], the modified simple equation method [3-6], the tanh-function method [7,8], the Exp-function method [9-13], the Jacobi elliptic function method [14], the (G'/G)-expansion method [15-23], the homotopy perturbation method [24,25], the transformed rational function method [26], the Ricatti ansätze [27], the multiple exp-function method [28,29], the generalize Hirota bilinear method [30], the Frobenius Integrable Decompositions [31] and so on.

Among those approaches, an enhanced (G'/G)-expansion method is a tool to reveal the solitons and periodic wave solutions of NLEEs in mathematical physics and engineering. The main ideas of the enhanced (G'/G)-expansion method are that the traveling wave solutions of NLEEs can be expressed as rational functions of (G'/G), where $G = G(\xi)$ satisfies the second order linear ordinary differential equation $G'' + \mu G = 0$. From which we conclude that the enhanced (G'/G)-expansion method is a particular case of the transformed rational function method [26], is almost similar to that of ricatti ansätze [27], and also like the Frobenius' idea [31].

The objective of this article is to present an enhanced (G'/G)-expansion method to construct the exact solitary wave solutions for NLEEs in mathematical physics via the YTSF equation.

The article is arranged as follows: In section 2, the enhanced (G'/G)-expansion method is discussed. In section 3, we apply this method to the nonlinear evolution equations pointed out above; in section 4, physical explanation; in section 5 comparisons and in section 6 conclusions are given.

2. MATERIAL AND METHOD

In this section, we describe the proposed enhanced (G'/G)-expansion method for finding traveling wave solutions of NLEEs. Suppose that a nonlinear partial differential equation, say in two independent variables x and t is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \cdots\cdots\cdots\cdots) = 0\,, \tag{2.1}$$

where $u(\xi) = u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this proposed method:

Step 1. Combining the independent variables x and t into one variable $\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x,t)\,, \qquad \xi = x \pm \omega t\;. \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ODE:

$$\wp(u, u', u'', \cdots\cdots\cdots) = 0\,, \tag{2.3}$$

where $\wp$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \dfrac{du}{d\xi}$, $u''(\xi) = \dfrac{d^2u}{d\xi^2}$ and so on.

Step 2.We suppose that Eq.(2.3) has the formal solution

$$u(\xi) = \sum_{i=-n}^{n}\left(\frac{a_i (G'/G)^i}{\left(1+\lambda(G'/G)\right)^i} + b_i (G'/G)^{i-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}\right), \tag{2.4}$$

where $G = G(\xi)$ satisfy the equation $G'' + \mu G = 0$, (2.5)

in which $a_i, b_i\ (-n \le i \le n; n \in \mathrm{N})$ and λ are constants to be determined later, and $\sigma = \pm 1, \mu \ne 0$.

Step 3. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(2.1) or Eq.(2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\; D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n+q)\,. \tag{2.6}$$

Therefore we can find the value of n in Eq.(2.4), using Eq.(2.6).

Step 4. We substitute Eq. (2.4) into Eq.(2.3) using Eq. (2.5) and then collect all terms of same powers of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}$ together, then set each coefficient of them to zero to yield a over-determined system of algebraic equations, solve this system for a_i, b_i, λ and ω.

Step 5. From the general solution of Eq.(2.5), we get
When $\mu < 0$,

$$\frac{G'}{G} = \sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi) \tag{2.7}$$

And

$$\frac{G'}{G} = \sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi) \tag{2.8}$$

Again, when $\mu > 0$,

$$\frac{G'}{G} = \sqrt{\mu}\tan(A-\sqrt{\mu}\xi) \tag{2.9}$$

And

$$\frac{G'}{G} = \sqrt{\mu}\cot(A+\sqrt{\mu}\xi) \tag{2.10}$$

where A is an arbitrary constant. Finally, substituting $a_i, b_i\ (-n \le i \le n; n \in \mathrm{N})$, λ, ω and Eqs. (2.7)-(2.10) into Eq. (2.4) we obtain traveling wave solutions of Eq. (2.1).

3. APPLICATION

In this section, we will exert enhanced (G'/G)-expansion method to solve the YTSF equation in the form,

$$-4u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx}u_z + 3u_{yy} = 0\ , \tag{3.1}$$

where $u(x, y, z, t)$ is the amplitude of the relative wave mode.

The traveling wave transformation equation $u(x, y, z, t) = u(\xi)$, $\xi = x + y + z + \omega t$ transform Eq.(3.1) to the following ordinary differential equation:

$$-4\omega u'' + u^{iv} + 6u''u' + 3u'' = 0. \tag{3.2}$$

Now integrating Eq. (3.2) with respect to ξ once, we have

$$u''' + 3(u')^2 + (3-4\omega)u' + R = 0. \tag{3.3}$$

where R is a constant of integration. Balancing the highest-order derivative term u''' and the nonlinear term $(u')^2$ from Eq.(3.3), yields $2(n+1) = n+3$ which gives $n = 1$.

Hence for $n = 1$ Eq.(2.4) reduces to

$$u(\xi) = \frac{a_{-1}(1+\lambda(G'/G))}{(G'/G)} + a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)} \\ + b_0(G'/G)^{-1}\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)} + b_1\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}, \quad (3.4)$$

where $G = G(\xi)$ satisfies Eq.(2.5). Substitute Eq.(3.4) along with Eq.(2.5) into Eq.(3.3). As a result of this substitution, we get a polynomial of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}$. From these polynomials, we equate the coefficients of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{1}{\mu}(G'/G)^2\right)}$, and setting them to zero, we get an over-determined system that consists of twenty-five algebraic equations. Solving this system for a_i, b_i, λ and ω ,we obtain the following values with the aid of symbolic computer software Maple 13.

Case 1: $R = 0, \omega = \frac{3}{4} - \mu, \lambda = \lambda, a_{-1} = 0, a_0 = a_0, a_1 = 2(1+\mu\lambda^2), b_{-1} = 0, b_0 = 0, b_1 = 0$.

Case 2: $R = 0, \omega = \frac{1}{4}(3-\mu), \lambda = 0, a_{-1} = 0, a_0 = a_0, a_1 = 1, b_{-1} = 0, b_0 = 0, b_1 = \pm\sqrt{\left(\frac{\mu}{\sigma}\right)}$.

Case 3 : $R = 0, \omega = \frac{3}{4} - \mu, \lambda = \lambda, a_{-1} = -2\mu, a_0 = a_0, a_1 = 0, b_{-1} = 0, b_0 = 0, b_1 = 0$.

Case 4: $R = 0, \omega = \frac{3}{4} - 4\mu, \lambda = 0, a_{-1} = -2\mu, a_0 = a_0, a_1 = 2, b_{-1} = 0, b_0 = 0, b_1 = 0$.

Case 5: $R = 0, \omega = \frac{1}{4}(3-\mu), \lambda = \lambda, a_{-1} = -\mu, a_0 = a_0, a_1 = 0, b_{-1} = 0,$

$b_0 = \pm\mu\sqrt{\left(\frac{1}{\sigma}\right)}, b_1 = 0$.

Hyperbolic function solutions: Substituting Eq. (2.7) and Eq. (2.8) into Eq. (3.4) along with Case 1-Case 5, we get the following five families of hyperbolic function solutions respectively.

Family 1: $$u_1(\xi) = a_0 + 2\sqrt{-\mu}\,(1+\mu\lambda^2)\left(\frac{\tanh(A+\sqrt{-\mu}\xi)}{1+\lambda\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi)}\right),$$

$$u_2(\xi) = a_0 + 2\sqrt{-\mu}\,(1+\mu\lambda^2)\left(\frac{\coth(A+\sqrt{-\mu}\xi)}{1+\lambda\sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi)}\right),$$

where $\xi = x + y + z + \left(\frac{3}{4} - \mu\right)t$.

Family 2: $$u_3(\xi) = a_0 + \sqrt{-\mu}\left(\tanh(A+\sqrt{-\mu}\xi) \pm I\,\mathrm{sec}h(A+\sqrt{-\mu}\xi)\right),$$

$$u_4(\xi) = a_0 + \sqrt{-\mu}\left(\coth(A+\sqrt{-\mu}\xi) \pm \mathrm{csc}h(A+\sqrt{-\mu}\xi)\right),$$

where $\xi = x + y + z + \frac{1}{4}(3-\mu)t$.

Family 3: $$u_5(\xi) = a_0 + 2\sqrt{-\mu}\left(\lambda\sqrt{-\mu} + \coth(A+\sqrt{-\mu}\xi)\right),$$

$$u_6(\xi) = a_0 + 2\sqrt{-\mu}\left(\lambda\sqrt{-\mu} + \tanh(A+\sqrt{-\mu}\xi)\right),$$

where $\xi = x + y + z + \left(\frac{3}{4} - \mu\right)t$.

Family 4: $$u_7(\xi) = a_0 + 2\sqrt{-\mu}\left(\tanh(A+\sqrt{-\mu}\xi) + \coth(A+\sqrt{-\mu}\xi)\right),$$

where $\xi = x + y + z + \left(\frac{3}{4} - 4\mu\right)t$.

Family 5: $$u_8(\xi) = a_0 + \sqrt{-\mu}\left(\coth(A+\sqrt{-\mu}\xi) \mp \mathrm{csc}h(A+\sqrt{-\mu}\xi) + \lambda\sqrt{-\mu}\right),$$

$$u_9(\xi) = a_0 + \sqrt{-\mu}\left(\tanh(A+\sqrt{-\mu}\xi) \mp I\,\mathrm{sec}h(A+\sqrt{-\mu}\xi) + \lambda\sqrt{-\mu}\right),$$

where $\xi = x + y + z + \frac{1}{4}(3-\mu)t$.

Trigonometric function solutions: Substituting Eq. (2.9) and Eq. (2.10) into Eq. (3.4) along with Case 1-Case 5, we get the following five trigonometric function solutions respectively.

Family 6: $$u_{10}(\xi) = a_0 + 2\sqrt{\mu}\,(1+\mu\lambda^2)\left(\frac{\tan(A-\sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}\tan(A-\sqrt{\mu}\xi)}\right),$$

$$u_{11}(\xi) = a_0 + 2\sqrt{\mu}\,(1+\mu\lambda^2)\left(\frac{\coth(A+\sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}\coth(A+\sqrt{\mu}\xi)}\right),$$

where $\xi = x + y + z + \left(\frac{3}{4} - \mu\right)t$.

Family 7: $$u_{12}(\xi) = a_0 + \sqrt{\mu}\left(\tan(A-\sqrt{\mu}\xi) \pm \sec(A-\sqrt{\mu}\xi)\right),$$

$$u_{13}(\xi) = a_0 + \sqrt{\mu}\left(\cot(A+\sqrt{\mu}\xi) \pm \csc(A+\sqrt{\mu}\xi)\right),$$

where $\xi = x + y + z + \frac{1}{4}(3 - \mu)t$.

Family 8: $u_{14}(\xi) = a_0 + 2\sqrt{\mu}\left(\lambda\sqrt{\mu} + \cot(A - \sqrt{\mu}\xi)\right),$

$u_{15}(\xi) = a_0 + 2\sqrt{\mu}\left(\lambda\sqrt{\mu} + \tan(A + \sqrt{\mu}\xi)\right),$

where $\xi = x + y + z + \left(\frac{3}{4} - \mu\right)t$.

Family 9: $u_{16}(\xi) = a_0 + 2\sqrt{\mu}\left(\tan(A - \sqrt{\mu}\xi) - \cot(A - \sqrt{\mu}\xi)\right),$

where $\xi = x + y + z + \left(\frac{3}{4} - 4\mu\right)t$.

Family 10: $u_{17}(\xi) = a_0 - \sqrt{\mu}\left(\cot(A - \sqrt{\mu}\xi) \mp \csc(A - \sqrt{\mu}\xi) + \lambda\sqrt{\mu}\right),$

$u_{18}(\xi) = a_0 - \sqrt{\mu}\left(\tan(A + \sqrt{\mu}\xi) \mp \sec(A + \sqrt{\mu}\xi) + \lambda\sqrt{\mu}\right),$

where $\xi = x + y + z + \frac{1}{4}(3 - \mu)t$.

4. PHYSICAL EXPLANATION

4.1 Results and Discussion

In this sub-section, we will discuss about the desired solutions of YTSF equation. It is interesting to point out that the delicate balance between the nonlinearity effect and the linear effect gives rise to solitons, that after a fully interaction with others, the solitons come back retaining their identities with the same speed and shape. If two solitons collide, then these just pass through each other and emerge unchanged.

When $\mu < 0$, $u_1(\xi)$ - $u_9(\xi)$ are exact traveling wave solutions of YTSF equation. For special values of the parameters solitary wave solutions are originated from these exact solutions.

- For the particular values of $\mu = -1,\ \lambda = 2, a_0 = 1, A = 2, y = z = 0$; $\mu = -1, a_0 = 1, A = 2, y = z = 0$ and $\mu = -1,\ \lambda = 1, a_0 = 1, A = 0, y = z = 0$ within the interval $-10 \le x, t \le 10$, $u_2(\xi)$, $u_3(\xi)$ and $u_8(\xi)$ are kink waves represented in Fig. 1, Fig. 2 and Fig. 5 respectively.
- For the particular values of $\mu = -1,\ \lambda = 2, a_0 = 1, A = -1, y = z = 0$ within the interval $-10 \le x, t \le 10$, $u_5(\xi)$ is soliton, represented in Fig. 3.
- For the particular values of $\mu = -1,\ a_0 = 1, A = 0, y = z = 0$ within the interval $-10 \le x, t \le 10$, $u_7(\xi)$ is a singular soliton represented in Fig. 4.

Consequently, for $\mu > 0$, Family 6-Family 10 are trigonometric function solutions, also said to be plane periodic traveling wave solutions.

- For the values of $\mu = 1,\ \lambda = 1, a_0 = 1, A = 1, y = z = 0$; $\mu = 1, a_0 = 1, A = 1, y = z = 0$; $\mu = 2,\ \lambda = 1, a_0 = 0, A = 0, y = z = 0$; $\mu = 1,$ $a_0 = 1, A = 0, y = z = 0$ and $\mu = 1, \lambda = 1, a_0 = 1, A = 0, y = z = 0$ within the interval $-10 \le x, t \le 10$, $u_{11}(\xi)$, $u_{12}(\xi)$, $u_{14}(\xi)$, $u_{16}(\xi)$ and $u_{17}(\xi)$ provides periodic wave solutions, which are represented in Fig. 6, Fig. 7, Fig. 8, Fig. 9 and Fig. 10 respectively.

  The wave speed ω plays an important role in the physical structure of the solutions obtained above. For the positive values of wave speed ω the disturbance represented by $u(\xi) = u(x - \omega t)$ are moving in the positive x-direction. Consequently, the negative values of wave speed ω the disturbance represented by $u(\xi) = u(x - \omega t)$ are moving in the negative x–direction.

4.2 Graphical Representation

Some of our obtained traveling wave solutions are represented in the following Figs.:

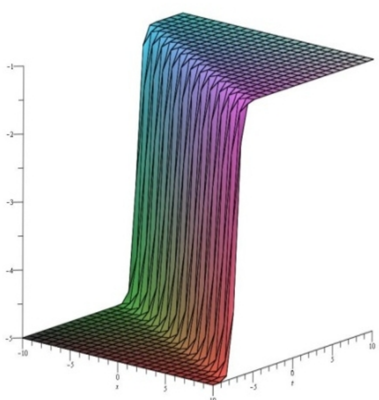

Fig. 1. Shape of $u_2(\xi)$ for
$\mu = -1,\ \lambda = 2, a_0 = 1, A = 2, y = z = 0$.

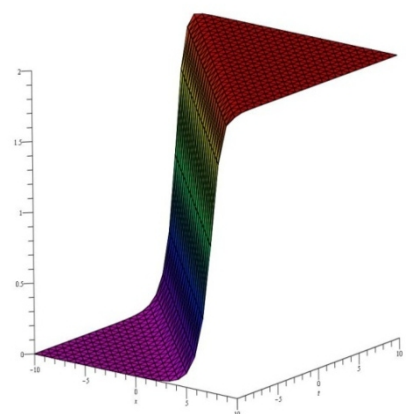

Fig. 2. Profile of $u_3(\xi)$ for
$\mu = -1, a_0 = 1, A = 2, y = z = 0$.

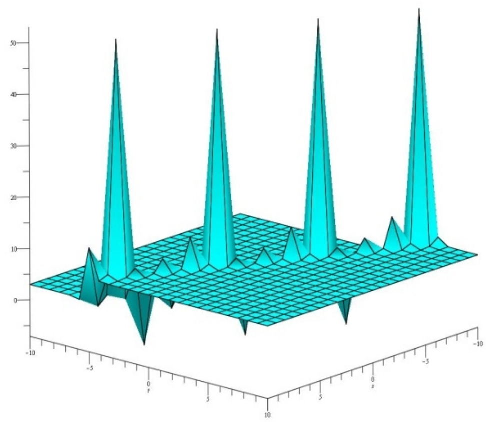

Fig. 3. Profile of $u_5(\xi)$ **for**
$\mu=-1,\ \lambda=2, a_0=1, A=-1, y=z=0$.

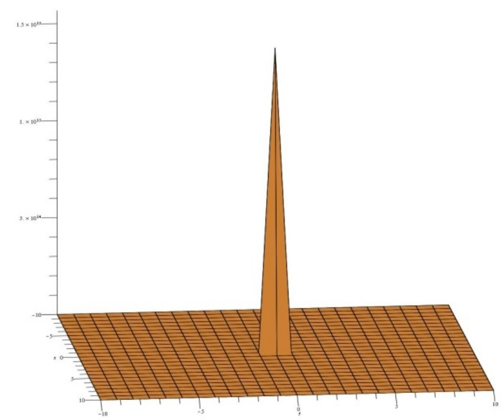

Fig. 4. Profile of $u_7(\xi)$ **for**
$\mu=-1,\ a_0=1, A=0, y=z=0$.

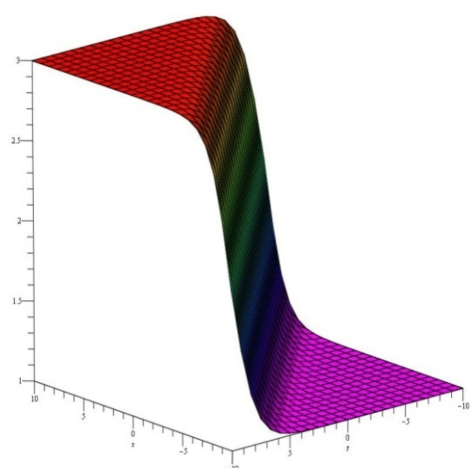

Fig. 5. Profile of $u_8(\xi)$ **for**
$\mu=-1,\ \lambda=1, a_0=1, A=0, y=z=0$.

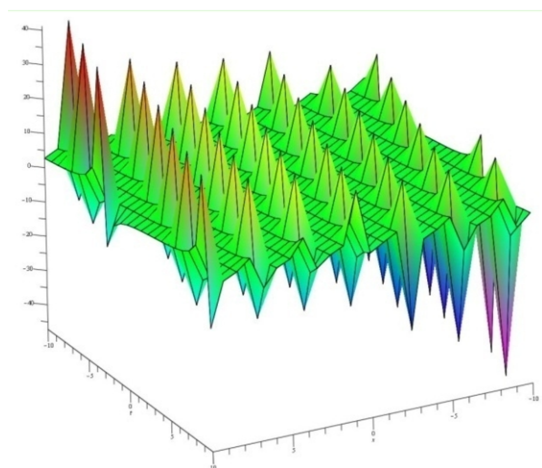

Fig. 6. Shape of $u_{11}(\xi)$ **for**
$\mu=1,\ \lambda=1, a_0=1, A=1, y=z=0$.

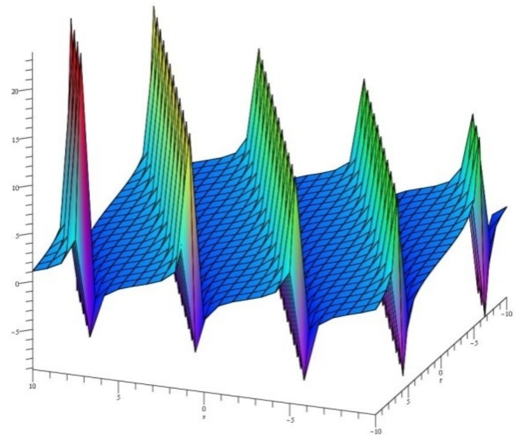

Fig. 7. Profile of $u_{12}(\xi)$ for
$\mu = 1, a_0 = 1, A = 1, y = z = 0$.

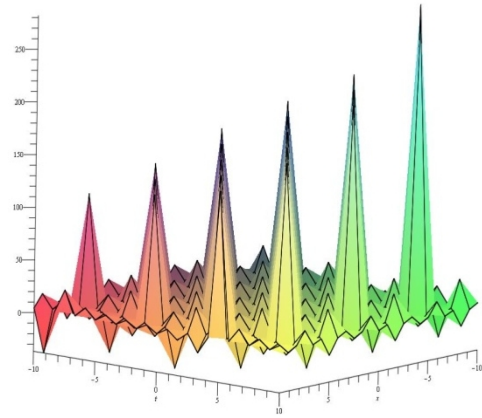

Fig. 8. Profile of $u_{14}(\xi)$ for
$\mu = 2,\ \lambda = 1, a_0 = 0, A = 0, y = z = 0$.

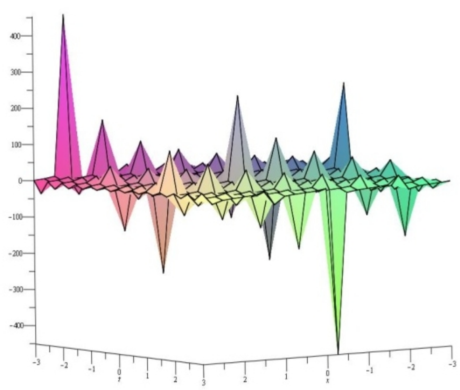

Fig. 9. Profile of $u_{16}(\xi)$ for
$\mu = 1,\ a_0 = 1, A = 0, y = z = 0$.

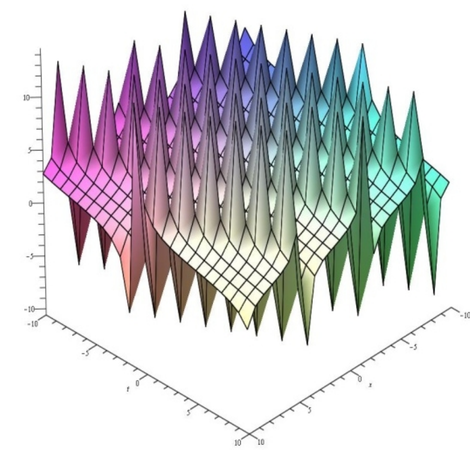

Fig. 10. Profile of $u_{17}(\xi)$ for
$\mu = 1,\ \lambda = 1, a_0 = 1, A = 0, y = z = 0$.

5. COMPARISONS

A. With modified simple equation method: Zayed and Arnous [6] investigated exact solutions of the Potential YTSF equation by using the modified simple equation method and obtained only one solution (**see APPENDIX A**). On the contrary by using the enhanced (G'/G)-expansion method in this article we obtained eighteen solutions. Furthermore, If we set $\mu = \dfrac{3l^2 + 4c}{4m}$, $\lambda = -\dfrac{4m}{2(3l^2 + 4c)}$, $A = \sqrt{\dfrac{-(3l^2 + 4c)}{4m}}\xi_0$ and $\xi = x + l\,y + m\,z - c\,t$ in our solution $u_6(\xi)$ (in Family 3) , we conclude that our result is equivalent to the result obtained by Zayed and Arnous [6].

B. With (G'/G) -expansion method: Zayed [23] examined exact solutions of the Potential YTSF equation by using the (G'/G)-expansion method and obtained three solutions (**see APPENDIX B**). On the contrary by using the enhanced (G'/G)-expansion method in this article we obtained eighteen solutions. Furthermore, If we set $\lambda = 0$ then our solutions $u_1(\xi)$, $u_2(\xi)$ (Family 1) and $u_5(\xi), u_6(\xi)$ (Family 3) coincide with the solution Eq. (B.3) obtained by Zayed [22] for $\lambda = 0$, $A = \sinh A$, $B = \cosh A$ and for $\lambda = 0$, $A = \cosh A$, $B = \sinh A$. Correspondingly, for similar conditions our solutions of Family 6 and Family 8 coincide with the solution Eq. (B.4) obtained by Zayed [23].

C. With Exp-function Method: Borhanifar and Kabir [13] investigated exact solutions of the Potential YTSF equation by using the Exp-function method and obtained the solutions (22) and (23) (**see APPENDIX C**).If we set $a_1 - k = a_0,\ k = l = s = \sqrt{-\mu}$ into Eq. (23) obtained by Borhanifar and Kabir [13] and $A = 0$ in our solution $u_3(\xi)$, we observe that our solution $u_3(\xi)$ coincides with the solution Eq.(23) obtained by Borhanifar and Kabir [13]. Similarly, If we set $a_1 - ki = a_0, k = l = s = \sqrt{\mu}$ into Eq. (22) obtained by Borhanifar and Kabir [13] and $A = \lambda = 0$ in our solution $u_{18}(\xi)$, we observe that our solution $u_{18}(\xi)$ coincides with the solution Eq.(22) obtained by Borhanifar and Kabir [13].

D. With multiple exp-function method: Ma et al. [28] investigated exact solutions of the Potential YTSF equation by using the multiple Exp-function method and obtained one-wave solutions(**see APPENDIX D**), two wave solutions and three wave solutions. If we set $b_0 = b_1 = 2,\ a_0 = 2k_1,\ k_1 = l_1 = m_1 = 2\sqrt{-\mu}$ into Eq. (3.5) obtained by Ma et al. [28] and $A = \lambda = 0$ in our solution $u_6(\xi)$, we observe that our solution $u_6(\xi)$ coincides with the solution Eq.(3.5) obtained by Ma et al. [28].

Similarly, If we set $b_0 = -b_1 = 2, a_0 = 2k_1,\ k_1 = l_1 = m_1 = 2\sqrt{-\mu}$ into Eq. (3.5) obtained by Ma et al. [28] and $A = \lambda = 0$ in our solution $u_5(\xi)$, we observe that our solution $u_5(\xi)$ coincides with the solution Eq.(3.5) obtained by Ma et al. [28].

6. CONCLUSIONS

In this paper, an enhanced (G'/G)-expansion method has been successfully applied to find the solitary wave solutions for the Potential YTSF equation. An abundant sets of solutions, of a variety of distinct physical structures such as solitons, singular solitons and periodic solutions were formally derived. The study highlights the power of these methods for the determination of exact solutions to several nonlinear evolution equations.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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APPENDIX A

Zayed and Arnous [6] examined the exact solutions of the Potential YTSF equation by making use the modified simple equation method. They assumed the solution is of the form,

$u(\xi)=\sum_{k=0}^{N} A_k\left(\frac{\psi'}{\psi}\right)^k$ and they obtained the only one solution,

$$u(\xi)=\left(A_0+1\right)+2\sqrt{\frac{-\left(3l^2+4c\right)}{4m}}\tanh\left(\sqrt{\frac{-\left(3l^2+4c\right)}{4m}}(\xi+\xi_0)\right). \qquad \text{(A.1)}$$

APPENDIX B

Zayed [23] examined the exact solutions of the Potential YTSF equation by using the (G'/G)-expansion method. He assumed the solution is of the form,

$$u(\xi)=\sum_{i=0}^{n}\alpha_i\left(\frac{G'}{G}\right)^i, \tag{B.1}$$

where $\xi = x + y + z - Vt$ and $G = G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0, \tag{B.2}$$

where α_i, V, λ and μ are constants to be determined later provided $\alpha_n \neq 0$.
By using the (G'/G)-expansion method Zayed [23] obtained the following three types of traveling wave solutions:

Case 1. If $\lambda^2 - 4\mu > 0$, then we have

$$u(\xi)=\sqrt{\lambda^2-4\mu}\left(\frac{A\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)+B\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}{A\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)+B\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}\right)+\alpha_0-\lambda, \tag{B.3}$$

Case 2. If $\lambda^2 - 4\mu < 0$, then we have

$$u(\xi)=\sqrt{4\mu-\lambda^2}\left(\frac{-A\sinh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)+B\cosh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)}{A\cosh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)+B\sinh\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi\right)}\right)+\alpha_0-\lambda, \tag{B.4}$$

Case 3. If $\lambda^2 - 4\mu = 0$, then we have

$$u(\xi)=2\left(\frac{B}{A+B\xi}\right)+\alpha_0-\lambda. \tag{B.5}$$

In particular, If $A=0, B\neq 0, \lambda>0, \mu=0$, then we deduce from (B.3) that

$$u(\xi)=\lambda\tanh\left(\frac{\lambda}{2}\xi\right)+\alpha_0-\lambda. \tag{B.6}$$

APPENDIX C

Borhanifar and Kabir [13] examined the exact solutions of the Potential YTSF equation by using Exp-function method and found the following solutions:

$$\begin{aligned} u(x,y,z,t) = a_1 - k\,i \pm k \sec\left(kx + ly + sz + \frac{3l^2 - k^3 s}{4k} t\right) \\ - k \tan\left(kx + ly + sz + \frac{3l^2 - k^3 s}{4k} t\right) \end{aligned} . \quad (22)$$

and

$$\begin{aligned} u(x,y,z,t) = (a_1 - k) \mp k\,i\ \sec h\left(kx + ly + sz + \frac{3l^2 + k^3 s}{4k} t\right) \\ + k \tanh\left(kx + ly + sz + \frac{3l^2 + k^3 s}{4k} t\right) \end{aligned} . \quad (23)$$

APPENDIX D

Ma et al. [28] examined the exact solutions of the Potential YTSF equation by using multiple exp-function method and found the following one wave solution solutions:

$$u(x,y,z,t)=\frac{a_0+a_1e^{k_1x+l_1y+m_1z-\omega_1t}}{b_0+b_1e^{k_1x+l_1y+m_1z-\omega_1t}}, \tag{3.5}$$

where $a_1=\dfrac{b_1(2k_1b_0+a_0)}{b_0}$ and $\omega_1=-\dfrac{1}{4}k_1^{\,2}m_1-\dfrac{3}{4}\dfrac{l_1^{\,2}}{k_1}$.

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