Asian Journal of Probability and Statistics

Volume 26, Issue 8, Page 89-106, 2024; Article no.AJPAS.117820
ISSN: 2582-0230

Discrete Erlang Mixed Distributions and their Properties

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: https://doi.org/10.9734/ajpas/2024/v26i8639

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/117820

Received: 14/05/2024
Accepted: 18/07/2024
Published: 26/07/2024

Original Research Article

Abstract

The proposed research is on discrete Erlang mixtures. Properties of the mixed distributions analyzed include raw and central moments, which have been derived in terms of moments of the mixing distributions. Cumulants obtained from the cumulant generating functions were also used in deriving the moments. The posterior distribution and posterior moments are also among properties presented. Bayesian, moments and maximum likelihood methods have been applied in parameter estimation. Additionally, the

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Cite as: *Gathongo, Beatrice M. 2024. "Discrete Erlang Mixed Distributions and Their Properties". Asian Journal of Probability and Statistics 26 (8):89-106. https://doi.org/10.9734/ajpas/2024/v26i8639.*

mixture distributions have been fitted to two data sets to test their goodness of fit. Some methods and special functions used in the study are the exponential series, logarithmic series, geometric series, modified Bessel function of the first kind, and the Touchard polynomials. The discrete mixing distributions used are the geometric, Poisson and logarithmic.

Keywords: Discrete Erlang mixtures; moments; cumulant; cumulant generating function; posterior distribution; Poisson; geometric; logarithmic.

1 Introduction

The Elang distribution is used in modeling the waiting time for an event in a Poisson process. It reduces to the exponential distribution when the shape parameter is equal to one. Its relation to both the Poisson and exponential distributions has contributed to its vast applications.

Mixed distributions are obtained by combining two or more distributions. They have a wider applicability compared to the basic distributions. They are used to model data that the basic distributions may fail to, and therefore are integral in situations that the basic distributions fail to address. They are devised by modifying the basic distributions using mixing weights to form finite mixtures, and by varying their shape parameters to form discrete mixtures and their rate/scale parameters to create continuous mixtures. Mixed Erlang distributions have been studied extensively over time. Zakerzadeh and Dolati [1], Shanker and Mishra [2], Merovci [3], Rashid et al. [4], Abouammoh et al. [5], Ghitany et al. [6], and Nadarajah et al. [7] are among people who derived finite Erlang mixtures, while McNolty [8], Jordanova and Stehlík [9], Jordanova et al. [10], and Kang [11] worked on continuous Erlang mixtures.

The focus of this work is on discrete Erlang mixed distributions, which are obtained by mixing the Erlang distribution with discrete mixing distributions. Tijms [12] showed that the Erlang mixture can be used in the approximation of any non-negative continuous distribution. Landriault et al. [13] evinced that the order statistics of independent mixed Erlang random variables belong to the same distribution class of Erlang mixtures. Cossette et al. [14] used mixtures of the Erlang distribution in moment based approximation. They conducted numerical experiments on the mixed Erlang approximation method, where the model was seen to provide an overall good fit. Woo [15] demonstrated that a large number of distributions are of the discrete mixed Erlang type. They showed that the Laplace transform of the Erlang mixture can be expressed in terms of the probability generating function of the mixing distribution. They also discussed special cases of the Erlang mixture, which include the exponential distribution, the Erlang distribution and the non-central chi-square distribution. Willmot and Woo [16] derived distributional properties of a class of multivariate mixed Erlang distributions with different scale parameters. Cossette et al. [17] presented the equilibrium function, among other properties, of the mixed Erlang distribution.

The outline of the paper is as follows: The mathematical formulation of the mixed Erlang distribution and its properties have been defined in section 2, and particular cases of the mixed distributions have been obtained in sections 3, 4 and 5 using the geometric, Poisson and logarithmic mixing distributions respectively. An application of the mixed distributions has been demonstrated in section 6 and section 7 contains the conclusion in brief.

2 The Discrete Mixed Erlang Distribution and Its Properties

- The probability density function (pdf) of the conditional (Erlang) distribution is

$$f(t|n) = \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}, \quad t > 0; \lambda > 0, n = 1, 2, 3, .. \tag{2.1}$$

and its distribution function (CDF) is given by

$$F(t|n) = 1 - e^{-\lambda t} \sum_{x=0}^{n-1} \frac{(\lambda t)^x}{x!} = \frac{\gamma(n, \lambda t)}{\Gamma(n)} \tag{2.2}$$

where n is the shape parameter and λ is the rate parameter.

- The mixed Erlang distribution is thus given by;

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} P_n = \lambda e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} P_n \\ &= \lambda e^{-\lambda t} E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) \end{aligned} \tag{2.3}$$

where P_n is a discrete mixing distribution.

- The r^{th} raw moment of the Erlang mixture is defined using conditional expectation as;

$$\begin{aligned} E(T^r) &= EE(T^r|n) = E\int_0^{\infty} t^r f(t|n) dt \\ &= E\int_0^{\infty} t^r \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1} dt = E\left(\frac{\lambda^n}{\Gamma n} \int_0^{\infty} t^{n+r-1} e^{-\lambda t} dt\right) \\ &= E\left(\frac{\lambda^n}{\Gamma n} \frac{\Gamma(n+r)}{\lambda^{n+r}}\right) = \frac{1}{\lambda^r} E\left(\frac{\Gamma(n+r)}{\Gamma n}\right) \end{aligned} \tag{2.4}$$

Raw and central moments of the mixed Erlang distribution in terms of moments of the mixing distribution are therefore given by;

- Raw moments

$$E(T) = \frac{1}{\lambda} E(n) \tag{2.5}$$

$$E(T^2) = \frac{1}{\lambda^2} E\left(\frac{\Gamma(n+2)}{\Gamma n}\right) = \frac{1}{\lambda^2} E[n(n+1)] = \frac{1}{\lambda^2}[E(n^2) + E(n)] \tag{2.6}$$

$$E(T^3) = \frac{1}{\lambda^3} E\left(\frac{\Gamma(n+3)}{\Gamma n}\right) = \frac{1}{\lambda^3} E[n(n+1)(n+2)] = \frac{1}{\lambda^3}[E(n^3) + 3E(n^2) + 2E(n)] \tag{2.7}$$

$$\begin{aligned} E(T^4) &= \frac{1}{\lambda^4} E\left(\frac{\Gamma(n+4)}{\Gamma n}\right) = \frac{1}{\lambda^4} E[n(n+1)(n+2)(n+3)] \\ &= \frac{1}{\lambda^4}[E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n)] \end{aligned} \tag{2.8}$$

- Central moments
  i. Variance

$$\begin{aligned} \mu_2 &= E\,[T - E(T)]^2 = E(T^2) - [E(T)]^2 \\ &= \frac{1}{\lambda^2}[E(n^2) + E(n)] - \frac{1}{\lambda^2}[E(n)]^2 = \frac{1}{\lambda^2}\left\{E(n^2) + E(n) - [E(n)]^2\right\} \\ &= \frac{1}{\lambda^2}\left\{Var(n) + E(n)\right\} \end{aligned} \tag{2.9}$$

ii. Third moment

$$
\begin{aligned}
\mu_3 &= E\left[T - E(T)\right]^3 = E(T^3) - 3E(T^2)E(T) + 2[E(T)]^3 \\
&= \frac{1}{\lambda^3}[E(n^3) + 3E(n^2) + 2E(n)] - \frac{3}{\lambda^3}[E(n^2) + E(n)]E(n) + \frac{2}{\lambda^3}[E(n)]^3 \\
&= \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \\
&= \frac{1}{\lambda^3}\left\{E[n - E(n)]^3 + 3Var(n) + 2E(n)\right\} \qquad (2.10)
\end{aligned}
$$

iii. Fourth moment

$$
\begin{aligned}
\mu_4 &= E\left[T - E(T)\right]^4 = E(T^4) - 4E(T^3)E(T) + 6E(T^2)[E(T)]^2 - 3[E(T)]^4 \\
&= \frac{1}{\lambda^4}[E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n)] - \frac{4}{\lambda^4}[E(n^3) + 3E(n^2) + \\
&\quad 2E(n)]E(n) + \frac{6}{\lambda^4}[E(n^2) + E(n)][E(n)]^2 - \frac{3}{\lambda^4}[E(n)]^4 \\
&= \frac{1}{\lambda^4}\{E(n^4) + 6E(n^3) + 11E(n^2) + 6E(n) - 4E(n^3)E(n) - 12E(n^2)E(n) - \\
&\quad 8[E(n)]^2 + 6E(n^2)[E(n)]^2 + 6[E(n)]^3 - 3[E(n)]^4\} \\
&= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + 6Var(n)E(n) + 11Var(n) + 3[E(n)]^2 + 6E(n)\} \\
&= \frac{1}{\lambda^4}\{E[n - E(n)]^4 + 6E[n - E(n)]^3 + Var(n)[6E(n) + 11] + 3[E(n)]^2 + 6E(n)\} \qquad (2.11)
\end{aligned}
$$

- The moment generating function of the Erlang mixture is given by

$$
\begin{aligned}
M_t(s) &= E(e^{ts}) = EE(e^{ts}|n) = E\int_0^\infty e^{ts} f(t|n)dt \\
&= E\int_0^\infty e^{ts}\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}dt = E\left(\frac{\lambda^n}{\Gamma n}\int_0^\infty t^{n-1}e^{-(\lambda - s)t}dt\right) \\
&= E\left(\frac{\lambda^n}{\Gamma n}\frac{\Gamma n}{(\lambda - s)^n}\right) = E\left(\frac{\lambda}{\lambda - s}\right)^n \qquad (2.12)
\end{aligned}
$$

and hence the cumulant generating function is

$$
K_t(s) = \mathsf{log}M_t(s) = \mathsf{log}E\left(\frac{\lambda}{\lambda - s}\right)^n \qquad (2.13)
$$

The r^{th} cumulant of the mixed distribution, $K_r(t)$, is the r^{th} derivative of the cumulant generating function at $s = 0$, and the first, second and third cumulants are the expected value, second and third central moments respectively.

$$K_t^{'}(s) = \frac{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]}{E\left(\frac{\lambda}{\lambda-s}\right)^n} \quad \text{and} \quad K_1(t) = K_t^{'}(0) = \frac{1}{\lambda}E(n) \tag{2.14}$$

$$K_t^{''}(s) = \frac{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left\{E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\right\}^2}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2} \quad \text{and}$$

$$K_2(t) = K_t^{''}(0) = \frac{1}{\lambda^2}\left\{E(n^2) + E(n) - [E(n)]^2\right\} \tag{2.15}$$

$$K_t^{'''}(s) = \frac{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)(n+2)\lambda^n}{(\lambda-s)^{n+3}}\right] - E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right]\right\} -}{\left[E\left(\frac{\lambda}{\lambda-s}\right)^n\right]^2}$$

$$\frac{2E\left[\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right]\left\{E\left(\frac{\lambda}{\lambda-s}\right)^n E\left[\frac{n(n+1)\lambda^n}{(\lambda-s)^{n+2}}\right] - \left[E\left(\frac{n\lambda^n}{(\lambda-s)^{n+1}}\right)\right]^2\right\}}{} \quad \text{and}$$

$$K_3(t) = K_t^{'''}(0) = \frac{1}{\lambda^3}\left\{E(n^3) + 3E(n^2) + 2E(n) - 3E(n^2)E(n) - 3[E(n)]^2 + 2[E(n)]^3\right\} \tag{2.16}$$

- The posterior distribution is given by

$$g(n|T) = \frac{f(t|n)P_n}{f(t)} = \frac{\frac{\lambda^n}{\Gamma n}e^{-\lambda t}t^{n-1}P_n}{\lambda e^{-\lambda t}E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.17}$$

where $f(t|n)$ is the likelihood function, which is the Erlang distribution, and P_n is the prior distribution. The posterior r^{th} moment is given by

$$E(n^r|T) = \sum_{n=1}^{\infty} n^r g(n|t) = \frac{\sum_{n=1}^{\infty} n^r \frac{(\lambda t)^{n-1}}{(n-1)!}P_n}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} = \frac{E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.18}$$

and the posterior mean is

$$E(n|T) = \frac{E\left(\frac{n(\lambda t)^{n-1}}{(n-1)!}\right)}{E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right)} \tag{2.19}$$

The posterior mean $E(n|T)$ is the Bayes estimator of the parameter n of the Erlang distribution, assuming squared error loss function.

Remark: The mixed Erlang distribution and its properties have been expressed in terms of expectations of the mixing distribution.

- Parameter estimation

Method of moments estimation (MME)

In this method, sample moments are used to estimate distribution moments, where the r^{th} raw moment for the sample $\underline{x}$ of size n,

$$m'_r = \frac{1}{n}\sum_{i=1}^{n} x_i^r, \quad r = 1, 2, 3, ... \tag{2.20}$$

is equated to the r^{th} raw moment of the probability distribution $f(x; \underline{\theta}), \underline{\theta} \in \Omega$,

$$\mu'_r = E(X^r), \quad r = 1, 2, 3, ... \tag{2.21}$$

to solve for the estimators of the parameters.

Maximum likelihood estimation (MLE)

The first derivatives of the log-likelihood functions of the mixed distributions, with respect to respective parameters, are obtained and equated to zero and the resulting equations solved simultaneously to obtain another equation, which is solved using the Newton-Raphson method to obtain the numerical estimates of the parameters.

3 Erlang-Geometric Distribution and Its Properties

The geometric mixing distribution is;

$$P_n = p(1-p)^{n-1}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{3.1}$$

$$\text{and,} \quad E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} p(1-p)^{n-1} = p\sum_{n=1}^{\infty} \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!} = pe^{\lambda t(1-p)} \tag{3.2}$$

$$\begin{aligned} \text{and,} \quad E\left(n^r \frac{(\lambda t)^{n-1}}{(n-1)!}\right) &= p\sum_{n=1}^{\infty} n^r \frac{[\lambda t(1-p)]^{n-1}}{(n-1)!} = \frac{p}{\lambda t(1-p)} \sum_{n=1}^{\infty} n^{r+1} \frac{[\lambda t(1-p)]^n}{n!} \\ &= \frac{p}{\lambda t(1-p)} e^{\lambda t(1-p)} T_{r+1}[\lambda t(1-p)] \end{aligned} \tag{3.3}$$

where $T_r(x) = e^{-x}\sum_{k=0}^{\infty} \frac{k^r x^k}{k!} = \sum_{k=0}^{r} S(r,k)x^k$ is the Touchard polynomials and $S(r,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j} j^r}{(k-j)!j!}$ is the Stirling number of the second kind.

a) The Erlang-geometric distribution is thus;

$$f(t) = \lambda p e^{-\lambda p t}, \quad t = 0, 1, 2, ...; 0 < p < 1, \lambda > 0 \tag{3.4}$$

The distribution function is given by

$$F(t) = \lambda p \sum_{x=0}^{t} e^{-\lambda p x} = \lambda p \left[\frac{1 - e^{-\lambda p(t+1)}}{1 - e^{-\lambda p}}\right] \tag{3.5}$$

and the quantile function is

$$Q(t) = F^{-1}(t) = \frac{1}{\lambda p}\ln\left[1 - \frac{t}{\lambda p}(1 - e^{-\lambda p})\right] - 1 \tag{3.6}$$

b) The moment generating function of the mixed distribution is;

$$\begin{aligned} M_t(s) &= \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda - s}\right)^n p(1-p)^{n-1} = \frac{p\lambda}{\lambda - s}\sum_{n=1}^{\infty}\left[\frac{\lambda(1-p)}{\lambda - s}\right]^{n-1} \\ &= \frac{p\lambda}{\lambda - s}\frac{1}{1 - \frac{\lambda(1-p)}{\lambda - s}} = \frac{p\lambda}{p\lambda - s} \end{aligned} \tag{3.7}$$

and the cumulant generating function is thus;

$$M_t(s) = \log\left(\frac{p\lambda}{p\lambda - s}\right) = \log(p\lambda) - \log(p\lambda - s) \tag{3.8}$$

c) The raw moments of the geometric distribution are;

$$E(n) = \frac{1}{p} \tag{3.9}$$

$$E(n^2) = \frac{2(1-p)}{p^2} + \frac{1}{p} \tag{3.10}$$

$$E(n^3) = \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} \tag{3.11}$$

$$E(n^4) = \frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} \tag{3.12}$$

and the central moments are therefore;

$$Var(n) = \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2} \tag{3.13}$$

$$E[n - E(n)]^3 = \frac{6(1-p)^2}{p^3} + \frac{6(1-p)}{p^2} + \frac{1}{p} - \frac{3(2-p)}{p^3} + \frac{2}{p^3} = \frac{2 - 3p + p^2}{p^3} \tag{3.14}$$

$$\begin{aligned} E[n - E(n)]^4 = &\frac{24(1-p)^3}{p^4} + \frac{36(1-p)^2}{p^3} + \frac{14(1-p)}{p^2} + \frac{1}{p} - \frac{4}{p^4}(6 - 12p + 6p^2 + 6p - 6p^2 + p^2) + \\ &\frac{6}{p^4}(2-p) - \frac{3}{p^4} = \frac{9 - 18p + 10p^2 - p^3}{p^4} \end{aligned} \tag{3.15}$$

d) Hence, the moments and cumulants of the Erlang-geometric distribution are given by;

$$E(T) = K_1(t) = \frac{1}{\lambda p} \tag{3.16}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}\left\{\frac{1-p}{p^2} + \frac{1}{p}\right\} = \frac{1}{(p\lambda)^2} \tag{3.17}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}\left\{\frac{2 - 3p + p^2}{p^3} + \frac{3(1-p)}{p^2} + \frac{2}{p}\right\} = \frac{2}{(p\lambda)^3} \tag{3.18}$$

$$\mu_4 = \frac{1}{\lambda^4}\left\{\frac{9 - 18p + 10p^2 - p^3}{p^4} + \frac{6(2 - 3p + p^2)}{p^3} + \frac{1-p}{p^2}\left[\frac{6}{p} + 11\right] + \frac{3}{p^2} + \frac{6}{p}\right\} = \frac{9}{(p\lambda)^4} \tag{3.19}$$

e) The posterior distribution of the mixed distribution is,

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!} p(1-p)^{n-1}}{pe^{\lambda t(1-p)}} = \frac{e^{-\lambda t(1-p)}[\lambda t(1-p)]^{n-1}}{(n-1)!} \tag{3.20}$$

which is Poisson$\sim [\lambda t(1-p)]$.

The posterior r^{th} moment is,

$$E(n^r|T) = \frac{\frac{p}{\lambda t(1-p)} e^{\lambda t(1-p)} T_{r+1}[\lambda t(1-p)]}{pe^{\lambda t(1-p)}} = \frac{T_{r+1}[\lambda t(1-p)]}{\lambda t(1-p)} \tag{3.21}$$

The posterior mean is hence given by,

$$E(n|T) = \frac{T_2[\lambda t(1-p)]}{\lambda t(1-p)} = \lambda t(1-p) + 1 \tag{3.22}$$

f) Parameter estimation

Method of moments

The method of moments estimator(MME) of the parameter p of the geometric distribution is

$$\frac{1}{p} = \bar{n} \implies \hat{p} = \frac{1}{\bar{n}} \tag{3.23}$$

and those of the parameters of the Erlang-geometric distribution are

$$\frac{1}{\lambda p} = \bar{t} \quad \text{and} \quad \frac{2}{(\lambda p)^2} = \frac{\sum_{i=1}^{n} t_i^2}{n} \implies \hat{\lambda} = \frac{1}{\hat{p}\bar{t}} \quad \text{and} \quad \hat{p} = \frac{1}{\hat{\lambda}\bar{t}} \tag{3.24}$$

The equation $\frac{1}{\lambda p} - \bar{t} = 0$ can be solved using the Newton-Raphson method to obtain the numerical estimates of $\hat{p}$ and $\hat{\lambda}$.

Maximum likelihood estimation

The likelihood function of the Erlang-geometric distribution is given by

$$L(p, \lambda) = \prod_{i=1}^{n} \lambda p e^{-\lambda p t_i} = (\lambda p)^n e^{-\lambda p \sum_{i=1}^{n} t_i} \tag{3.25}$$

and the log-likelihood function is thus

$$Ł = \ln L(p, \lambda) = n\ln(\lambda p) - \lambda p \sum_{i=1}^{n} t_i \tag{3.26}$$

The first derivatives with respect to respective parameters are obtained as illustrated below.

$$\frac{\delta Ł}{\delta p} = \frac{n}{p} - \lambda \sum_{i=1}^{n} t_i = 0 \tag{3.27}$$

$$\frac{\delta Ł}{\delta \lambda} = \frac{n}{\lambda} - p \sum_{i=1}^{n} t_i = 0 \tag{3.28}$$

$$\implies \hat{p} = \frac{n}{\hat{\lambda} \sum_{i=1}^{n} t_i} = \frac{1}{\hat{\lambda}\bar{t}} \quad \text{and} \quad \hat{\lambda} = \frac{n}{\hat{p} \sum_{i=1}^{n} t_i} = \frac{1}{\hat{p}\bar{t}} \implies \hat{p}\hat{\lambda} = \frac{1}{\bar{t}} \tag{3.29}$$

Equations (3.27)-(3.28) are equated to zero and solved simultaneously to further obtain the equation $\hat{p}\hat{\lambda} - \frac{1}{\bar{t}} = 0$, which is solved using the Newton-Raphson method to estimate the parameters p and λ numerically.

4 Erlang-Poisson Distribution and Its Properties

The Poisson mixing distribution is;

$$P_n = \frac{e^{-p}p^n}{n!}, \quad n = 0, 1, 2, ...; 0 < p < 1 \tag{4.1}$$

and, $$E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-p}p^n}{n!} = pe^{-p}\sum_{n=1}^{\infty}\frac{(\lambda pt)^{n-1}}{n!(n-1)!}\frac{(-1)^{n-1}}{(-1)^{n-1}}$$

$$= pe^{-p}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-\lambda pt)^{\frac{2(n-1)}{2}}}{n!(n-1)!} = \frac{pe^{-p}}{i\sqrt{\lambda pt}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\sqrt{-\lambda pt}\right)^{2n-1}}{n!(n-1)!}$$

$$= \frac{pe^{-p}}{i\sqrt{\lambda pt}}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{2i\sqrt{\lambda pt}}{2}\right)^{2n-1}}{n!(n-1)!} = \frac{pe^{-p}}{i\sqrt{\lambda pt}}\dot{\tau}_1\left(2i\sqrt{\lambda pt}\right) \tag{4.2}$$

where $\dot{\tau}_\rho(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\rho+1)}\left(\frac{x}{2}\right)^{2k+\rho}$ is the modified Bessel function of the first kind.

a) The Erlang-Poisson mixture is thus;

$$f(t) = \frac{\lambda pe^{-(\lambda t+p)}}{i\sqrt{\lambda pt}}\dot{\tau}_1(2i\sqrt{\lambda pt}), \quad t = 1, 2, 3, ...; 0 < p < 1, \lambda > 0 \tag{4.3}$$

and the distribution function is

$$F(t) = \frac{\lambda pe^{-p}}{\sqrt{-\lambda p}}\sum_{x=1}^{t}\frac{e^{-\lambda x}}{\sqrt{x}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+2)}(-\lambda px)^{k+\frac{1}{2}} = e^{-p}\sqrt{\lambda p}\sum_{k=0}^{\infty}\frac{(\lambda p)^{k+\frac{1}{2}}}{k!\Gamma(k+2)}\sum_{x=1}^{t}e^{-\lambda x}x^k$$

$$= e^{-p}\sqrt{\lambda p}\sum_{k=0}^{\infty}\frac{(\lambda p)^{k+\frac{1}{2}}}{k!\Gamma(k+2)}\left(e^{-z}\frac{d}{de^{-z}}\right)^k\frac{1-e^{-z(t+1)}}{1-e^{-z}} \tag{4.4}$$

b) The moment generating function of the mixture is;

$$M_t(s) = \sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda - s}\right)^n\frac{e^{-p}p^n}{n!} = e^{-p}\sum_{n=0}^{\infty}\left(\frac{\lambda p}{\lambda - s}\right)^n\frac{1}{n!}$$

$$= e^{-p\left(1-\frac{\lambda}{\lambda-s}\right)} = e^{\frac{ps}{\lambda-s}} \tag{4.5}$$

and the cumulant generating function is therefore;

$$K_t(s) = \ln\left(e^{\frac{ps}{\lambda-s}}\right) = \frac{ps}{\lambda - s} \tag{4.6}$$

c) The raw moments of the Poisson distribution are;

$$E(n) = p \tag{4.7}$$

$$E(n^2) = p^2 + p = p(p+1) \tag{4.8}$$

$$E(n^3) = p^3 + 3p^2 + p = p(p^2 + 3p + 1) \tag{4.9}$$

$$E(n^4) = p^4 + 6p^3 + 7p^2 + p = p(p^3 + 6p^2 + 7p + 1) \tag{4.10}$$

and the central moments are hence given by;

$$Var(n) = p^2 + p - p^2 = p \tag{4.11}$$

$$E[n - E(n)]^3 = p^3 + 3p^2 + p - 3p(p^2 + p) + 2p^3 = p \tag{4.12}$$

$$E[n - E(n)]^4 = p^4 + 6p^3 + 7p^2 + p - 4p(p^3 + 3p^2 + p) + 6p^2(p^2 + p) - 3p^4 = 3p^2 + p \tag{4.13}$$

d) Moments and cumulants of the Erlang-Poisson distribution are thus;

$$E(T) = K_1(t) = \frac{p}{\lambda} \tag{4.14}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}(p+p) = \frac{2p}{\lambda^2} \tag{4.15}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}(p+3p+2p) = \frac{6p}{\lambda^3} \tag{4.16}$$

$$\mu_4 = \frac{1}{\lambda^4}[3p^2+p+6p+p(6p+11)+3p^2+6p] = \frac{12}{\lambda^4}(p^2+2) \tag{4.17}$$

e) The posterior distribution of the mixture distribution is

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{e^{-p}p^n}{n!}}{\frac{pe^{-p}}{i\sqrt{\lambda pt}}\dot{\tau}_1\left(2i\sqrt{\lambda pt}\right)} = \frac{i\sqrt{\lambda pt}(\lambda pt)^{n-1}}{\dot{\tau}_1\left(2i\sqrt{\lambda pt}\right)n!(n-1)!} \tag{4.18}$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{i\sqrt{\lambda pt}}{\dot{\tau}_1\left(2i\sqrt{\lambda pt}\right)}\sum_{n=1}^{\infty} n^r \frac{(\lambda pt)^{n-1}}{n!(n-1)!} \tag{4.19}$$

and the posterior mean is

$$E(n|T) = \frac{i\sqrt{\lambda pt}}{\dot{\tau}_1\left(2i\sqrt{\lambda pt}\right)}\sum_{n=1}^{\infty}\frac{(\lambda pt)^{n-1}}{(n-1)!(n-1)!} = i\sqrt{\lambda pt}\frac{\dot{\tau}_0(2i\sqrt{\lambda pt})}{\dot{\tau}_1(2i\sqrt{\lambda pt})} \tag{4.20}$$

f) Parameter estimation

Method of moments

The respective method of moments estimators of the parameters of the Poisson and the Erlang-Poisson distributions are given by

$$\hat{p} = \bar{n} \tag{4.21}$$

$$\frac{p}{\lambda} = \bar{t} \quad \text{and} \quad \frac{p(2+p)}{\lambda^2} = \frac{\sum_{i=1}^n t_i^2}{n} \implies \frac{\bar{t}(2+\lambda\bar{t})}{\lambda} = \frac{\sum_{i=1}^n t_i^2}{n}$$

$$\implies \hat{\lambda} = \frac{2n\bar{t}}{\sum_{i=1}^n t_i^2 - n\bar{t}^2} \quad \text{and} \quad \hat{p} = \frac{2n\bar{t}^2}{\sum_{i=1}^n t_i^2 - n\bar{t}^2} \tag{4.22}$$

Maximum likelihood estimation

The likelihood function of the mixture is

$$L(p,\lambda) = \prod_{i=1}^{n}\frac{\lambda pe^{-(\lambda t_i+p)}}{\sqrt{-\lambda pt_i}}\dot{\tau}_1(2\sqrt{-\lambda pt_i}) = \frac{(\lambda p)^n e^{-np}e^{-\lambda\sum_{i=1}^n t_i}}{\left(\sqrt{-\lambda p}\right)^n\prod_{i=1}^n\sqrt{t_i}}\prod_{i=1}^{n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+2)}\left(\sqrt{-\lambda pt_i}\right)^{2k+1} \tag{4.23}$$

and the log-likelihood function is given by

$$\text{L} = \ln L(p,\lambda) = n\ln(\lambda p) - np - \lambda\sum_{i=1}^{n}t_i - n\ln\left(\sqrt{-\lambda p}\right) - \sum_{i=1}^{n}\ln\sqrt{t_i} +$$
$$\sum_{i=1}^{n}\ln\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+2)}\left(\sqrt{-\lambda pt_i}\right)^{2k+1} \tag{4.24}$$

The derivatives of the log-likelihood function with respect to the parameters, p and λ are obtained as below.

$$\frac{\delta \mathrm{L}}{\delta p} = \frac{n}{2p} - n + \lambda \sum_{i=1}^{n} \frac{\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+1)t_i}{2k!\Gamma(k+2)} \left(\sqrt{-\lambda p t_i}\right)^{2k-1}}{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2)} \left(\sqrt{-\lambda p t_i}\right)^{2k+1}} = 0 \tag{4.25}$$

$$\frac{\delta \mathrm{L}}{\delta \lambda} = \frac{n}{2\lambda} - \sum_{i=1}^{n} t_i + p \sum_{i=1}^{n} \frac{\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k+1)t_i}{2k!\Gamma(k+2)} \left(\sqrt{-\lambda p t_i}\right)^{2k-1}}{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+2)} \left(\sqrt{-\lambda p t_i}\right)^{2k+1}} = 0 \tag{4.26}$$

$$\implies \frac{\hat{p}}{\hat{\lambda}} = \frac{\sum_{i=1}^{n} t_i}{n} = \bar{t} \implies \hat{p} - \hat{\lambda}\bar{t} = 0 \tag{4.27}$$

The Newton-Raphson method is applied to the equation $\hat{p} - \hat{\lambda}\bar{t} = 0$ to obtain numerical maximum likelihood estimates of the parameters. The equation is obtained by equating equations (4.25)-(4.26) to zero and solving them simultaneously.

5 Erlang-Logarithmic Distribution and Its Properties

The logarithmic mixing distribution is given by;

$$P_n = \frac{p^n}{-n\log(1-p)}, \quad n = 1, 2, 3, ...; 0 < p < 1 \tag{5.1}$$

$$\text{and thus,} \quad E\left(\frac{(\lambda t)^{n-1}}{(n-1)!}\right) = \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \frac{p^n}{-n\log(1-p)} = \frac{1}{-\lambda t \log(1-p)} \sum_{n=1}^{\infty} \frac{(\lambda t p)^n}{n!}$$
$$= \frac{-(e^{\lambda t p} - 1)}{\lambda t \log(1-p)} = \frac{1 - e^{\lambda t p}}{\lambda t \log(1-p)} \tag{5.2}$$

$$\text{and,} \quad E\left(\frac{n^r(\lambda t)^{n-1}}{(n-1)!}\right) = \frac{1}{-\lambda t \log(1-p)} \sum_{n=1}^{\infty} \frac{n^r(\lambda t p)^n}{n!} = \frac{e^{\lambda t p} T_r(\lambda t p)}{-\lambda t \log(1-p)} \tag{5.3}$$

a) The Erlang-logarithmic distribution is therefore,

$$f(t) = \frac{e^{-\lambda t} - e^{-\lambda t(1-p)}}{t\log(1-p)}, \quad t = 1, 2, 3, ...; \lambda > 0, 0 < p < 1 \tag{5.4}$$

with a distribution function,

$$F(t) = \frac{1}{\log(1-p)} \left[\sum_{x=1}^{t} \frac{\left(e^{-\lambda}\right)^x}{x} - \sum_{x=1}^{t} \frac{\left(e^{-\lambda(1-p)}\right)^x}{x}\right]$$
$$= \frac{1}{\log(1-p)} \left[\log\left(\frac{1 - e^{-\lambda(1-p)}}{1 - e^{-\lambda}}\right) + B(e^{-\lambda(1-p)}; t+1, 0) - B(e^{-\lambda}; t+1, 0)\right] \tag{5.5}$$

b) The moment generating function of the mixed distribution is;

$$
\begin{aligned}
M_t(s) &= \sum_{n=1}^{\infty}\left(\frac{\lambda}{\lambda-s}\right)^n \frac{p^n}{-n\log(1-p)} = \frac{1}{-\log(1-p)}\sum_{n=1}^{\infty}\left(\frac{\lambda p}{\lambda-s}\right)^n \frac{1}{n} \\
&= \frac{\log\left[1-\frac{\lambda p}{\lambda-s}\right]}{\log(1-p)} = \frac{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)}{\log(1-p)}
\end{aligned} \tag{5.6}
$$

and hence the cumulant generating function is given by;

$$
K_t(s) = \log\left(\frac{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)}{\log(1-p)}\right) = \log\left\{\log\left[\lambda(1-p)-s\right]-\log(\lambda-s)\right\}-\log\log(1-p) \tag{5.7}
$$

c) The raw moments of the logarithmic distribution are;

$$
E(n) = \frac{-p}{(1-p)\log(1-p)} \tag{5.8}
$$

$$
E(n^2) = \frac{-p^2}{(1-p)^2\log(1-p)} - \frac{p}{(1-p)\log(1-p)} = \frac{-p}{(1-p)^2\log(1-p)} \tag{5.9}
$$

$$
E(n^3) = \frac{-2p^3}{(1-p)^3\log(1-p)} - \frac{3p^2}{(1-p)^2\log(1-p)} - \frac{p}{(1-p)\log(1-p)} = \frac{-p(p+1)}{(1-p)^3\log(1-p)} \tag{5.10}
$$

$$
\begin{aligned}
E(n^4) &= \frac{-6p^4}{(1-p)^4\log(1-p)} - \frac{12p^3}{(1-p)^3\log(1-p)} - \frac{7p^2}{(1-p)^2\log(1-p)} - \frac{p}{(1-p)\log(1-p)} \\
&= \frac{-p(p^2+4p+1)}{(1-p)^4\log(1-p)}
\end{aligned} \tag{5.11}
$$

and the central moments are;

$$
Var(n) = \frac{-p}{(1-p)^2\log(1-p)} - \frac{p^2}{(1-p)^2[\log(1-p)]^2} = \frac{-p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2} \tag{5.12}
$$

$$
\begin{aligned}
E\left[n-E(n)\right]^3 &= \frac{-p(p+1)}{(1-p)^3\log(1-p)} - \frac{3p^2}{(1-p)^3[\log(1-p)]^2} - \frac{2p^3}{(1-p)^3[\log(1-p)]^3} \\
&= \frac{-p(p+1)[\log(1-p)]^2-3p^2\log(1-p)-2p^3}{(1-p)^3[\log(1-p)]^3}
\end{aligned} \tag{5.13}
$$

$$
\begin{aligned}
E\left[n-E(n)\right]^4 &= \frac{-p(p^2+4p+1)}{(1-p)^4\log(1-p)} - \frac{4p^2(p+1)}{(1-p)^4[\log(1-p)]^2} - \frac{6p^3}{(1-p)^4[\log(1-p)]^3} - \\
&\quad \frac{3p^4}{(1-p)^4[\log(1-p)]^4} \\
&= \frac{-p(p^2+4p+1)[\log(1-p)]^3-4p^2(p+1)[\log(1-p)]^2-6p^3\log(1-p)-3p^4}{(1-p)^4[\log(1-p)]^4}
\end{aligned} \tag{5.14}
$$

d) Moments and cumulants of the Erlang-logarithmic distribution are therefore;

$$E(T) = K_1(t) = \frac{-p}{\lambda(1-p)\log(1-p)} \tag{5.15}$$

$$Var(T) = K_2(t) = \frac{1}{\lambda^2}\left\{\frac{-p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2} - \frac{p}{(1-p)\log(1-p)}\right\} = \frac{-p[p+(2-p)\log(1-p)]}{\lambda^2(1-p)^2[\log(1-p)]^2} \tag{5.16}$$

$$\mu_3 = K_3(t) = \frac{1}{\lambda^3}\left\{\frac{-p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{3p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2} - \right.$$
$$\frac{2p}{(1-p)\log(1-p)} = \frac{-2p^3 - p^2(4-p)\log(1-p) - p(6-6p+2p^2)[\log(1-p)]^2}{\lambda^3(1-p)^3[\log(1-p)]^3} \tag{5.17}$$

$$\mu_4 = \frac{1}{\lambda^4}\left\{\frac{-p(p^2+4p+1)[\log(1-p)]^3 - 4p^2(p+1)[\log(1-p)]^2 - 6p^3\log(1-p) - 3p^4}{(1-p)^4[\log(1-p)]^4} - \right.$$
$$\frac{6p(p+1)[\log(1-p)]^2 - 3p^2\log(1-p) - 2p^3}{(1-p)^3[\log(1-p)]^3} - \frac{p\log(1-p)-p^2}{(1-p)^2[\log(1-p)]^2}$$
$$\left[-\frac{6p}{(1-p)\log(1-p)} + 11\right] + \frac{3p^2}{(1-p)^2[\log(1-p)]^2} - \frac{6p}{(1-p)\log(1-p)}$$
$$= \frac{-6p(6p^2-6p-p^3+2)[\log(1-p)]^3 - 8p(3p^2+3p+p^3)[\log(1-p)]^2 -}{\lambda^4(1-p)^4[\log(1-p)]^4}$$
$$\underline{6p(2p^2-p^3)\log(1-p) - 3p^4} \tag{5.18}$$

e) The posterior distribution of the Erlang mixture is the zero truncated Poisson (λtp) distribution,

$$g(n|T) = \frac{\frac{(\lambda t)^{n-1}}{(n-1)!}\frac{p^n}{-n\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{(\lambda tp)^n}{n!(e^{\lambda tp}-1)} \tag{5.19}$$

The posterior r^{th} moment is

$$E(n^r|T) = \frac{\frac{e^{\lambda tp}T_r(\lambda tp)}{-\lambda t\log(1-p)}}{\frac{1-e^{\lambda tp}}{\lambda t\log(1-p)}} = \frac{T_r(\lambda tp)}{1-e^{-\lambda tp}} \tag{5.20}$$

and the posterior mean is

$$E(n|T) = \frac{T_1(\lambda tp)}{e^{\lambda tp}-1} = \frac{\lambda tp}{1-e^{-\lambda tp}} = \frac{(\lambda tp)e^{\lambda tp}}{e^{\lambda tp}-1} \tag{5.21}$$

f) Parameter estimation

Method of moments

The parameter p of the logarithmic distribution can be estimated, using method of moments, as

$$\frac{-p}{(1-p)\log(1-p)} = \bar{n} \implies \frac{\hat{p}}{(1-\hat{p})} + \bar{n}\log(1-\hat{p}) = 0 \tag{5.22}$$

The resulting equation can then be solved using the Newton-Raphson method, to estimate the parameter numerically.

The Erlang-logarithmic distribution method of moments parameter estimators, $\hat{p}$ and $\hat{\lambda}$ are given by

$$\frac{-p}{\lambda(1-p)\log(1-p)} = \bar{t} \quad \text{and} \quad \frac{-p(2-p)\log(1-p)}{\lambda^2(1-p)^2[\log(1-p)]^2} = \frac{\sum_{i=1}^{n} t_i^2}{n} \tag{5.23}$$

$$\implies n(2-\hat{p})\log(1-\hat{p})\bar{t}^2 + \hat{p}\sum_{i=1}^{n} t_i^2 = 0 \quad \text{and} \quad \hat{\lambda} = \frac{-\hat{p}}{\bar{t}(1-\hat{p})\log(1-\hat{p})} \tag{5.24}$$

The parameter p can be estimated numerically by solving the equation $n(2-\hat{p})\log(1-\hat{p})\bar{t}^2+\hat{p}\sum_{i=1}^{n} t_i^2 = 0$ using the Newton-Raphson method.

Maximum likelihood estimation

The likelihood function of the mixed distribution is

$$L(p,\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda t_i} - e^{-\lambda t_i(1-p)}}{t_i \log(1-p)} = \frac{\prod_{i=1}^{n}\left[e^{-\lambda t_i} - e^{-\lambda t_i(1-p)}\right]}{n\log(1-p)\prod_{i=1}^{n} t_i} \tag{5.25}$$

The log-likelihood function is

$$Ł = \log L(p,\lambda) = \sum_{i=1}^{n} \log\left[e^{-\lambda t_i} - e^{-\lambda t_i(1-p)}\right] - \log n - \log\log(1-p) - \sum_{i=1}^{n} \log t_i \tag{5.26}$$

The log-likelihood function is then differentiated with respect to the parameters, p and λ, resulting in the below equations, which are equated to zero and solved simultaneously. The Newton-Raphson method was applied in estimating the parameters numerically.

$$\frac{\delta Ł}{\delta p} = \sum_{i=1}^{n} \frac{-\lambda t_i e^{-\lambda t_i(1-p)}}{e^{-\lambda t_i} - e^{-\lambda t_i(1-p)}} + \frac{1}{(1-p)\log(1-p)} = \sum_{i=1}^{n} \frac{-\lambda t_i e^{-\lambda t_i p}}{1-e^{-\lambda t_i p}} + \frac{1}{(1-p)\log(1-p)} \tag{5.27}$$

$$\frac{\delta Ł}{\delta \lambda} = \sum_{i=1}^{n} \frac{-t_i e^{-\lambda t_i} + t_i(1-p)e^{-\lambda t_i(1-p)}}{e^{-\lambda t_i} - e^{-\lambda t_i(1-p)}} = \sum_{i=1}^{n} \frac{-t_i\left[1-(1-p)e^{-\lambda t_i p}\right]}{1-e^{-\lambda t_i p}} \tag{5.28}$$

6 Application

This section provides an application of the constructed distributions to data to assess and compare their goodness of fit. The -log-likelihood (-log(L)), chi-square (χ^2) and kolmogorov-smirnov (k-s) statistics have been computed to this effect. Two real data sets have been used. The first is on death times, in weeks, of 30 patients with cancer of the tongue. The data has been applied by various authors such as Klein et al. [18], Eledum and El-Alosey [19] and Eledum and El-Alosey [20]. The second data set is on remission times, in weeks, for 30 leukemia patients who are on a specific type of medication, and has been used by Eledum and El-Alosey [19], Eledum and El-Alosey [20] and Lawless [21] among other authors.

The mixed distributions generally offer better fits compared to the mixing distributions, as seen in Tables 3 and 4, where the p-values are smaller and the -log(L), χ^2 and k-s values are larger for mixing distributions compared to the corresponding mixed distributions. Among the mixed distributions, the Erlang-Poisson is a better fit compared to the Erlang-logarithmic mixture, but the Erlang-geometric distribution offers the best fit among the three mixtures with the least values for the -log(L), χ^2 and k-s and the highest p-value.

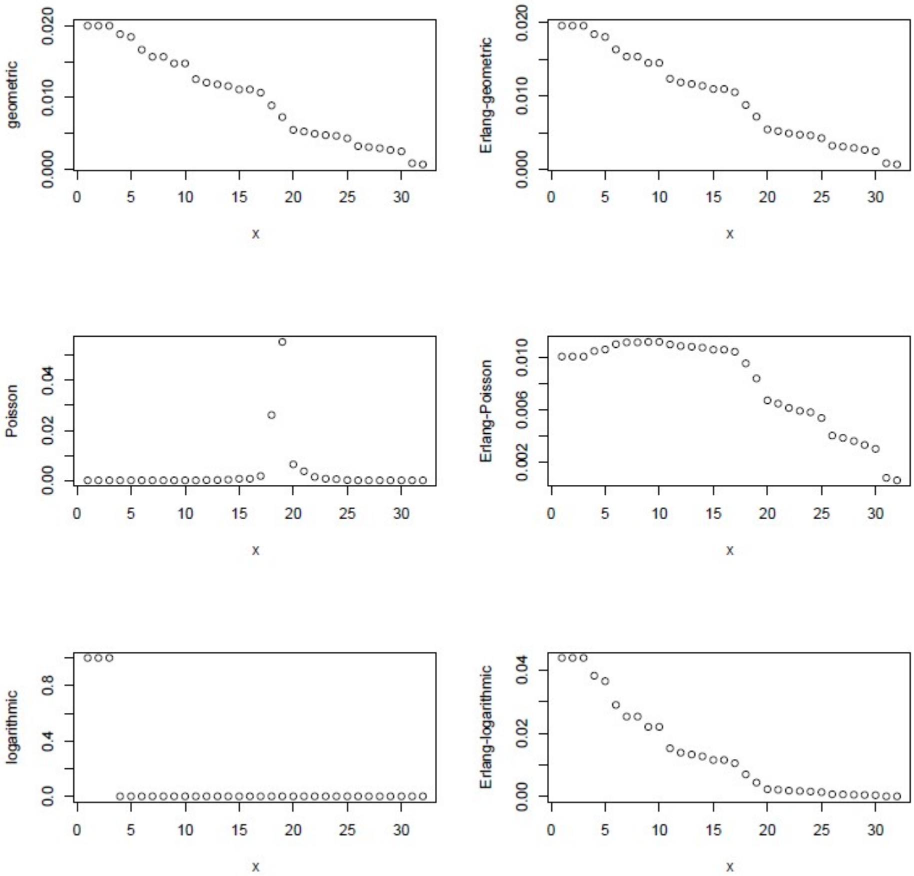

Fig. 1. The pmfs for the geometric, Erlang-geometric, Poisson, Erlang-Poisson, logarithmic and Erlang-logarithmic distributions using dataset 1

Table 1. Death times, in weeks, of patients with cancer of the tongue.

Dataset 1															
1	1	1	4	5	10	13	13	16	16	24	26	27	28	30	30
32	41	51	65	67	70	72	73	77	91	93	96	100	104	157	167

Table 2. Remission times, in weeks, for some leukemia patients taking a specific type of therapy.

Dataset 2														
1	1	2	4	4	6	6	6	7	8	9	9	10	12	13
14	18	19	24	26	29	31	42	45	50	57	60	71	85	91

Table 3. Parameters estimates, -log(L), χ^2, p-value and k-s test value for the mixing and mixed distributions using the tongue cancer patients' data set

Distribution	Estimated parameters		-log(L)	χ^2	p-value	k-s
geometric	$\hat{p}$=0.02	-	157	4	0.5	0.09
Poisson	$\hat{p}$=50.03	-	689.73	$\to \infty$	0	0.5
logarithmic	$\hat{p}$=0.0001	-	$\to \infty$	$\to \infty$	0	0.84
Erlang-geometric	$\hat{p}$=0.1	$\hat{\lambda}$=0.2	157	4	0.5	0.09
Erlang-Poisson	$\hat{p}$=2.66	$\hat{\lambda}$=0.053	160	8.29	0.14	0.164
Erlang-logarithmic	$\hat{p}$=0.0001	$\hat{\lambda}$=0.046	172.18	39	0.0000002	0.28

Table 4. Parameters estimates, -log(L), χ^2, p-value and k-s test value for the mixing and mixed distributions using the leukemia patients' data set

Distribution	Estimated parameters		-log(L)	χ^2	p-value	k-s
geometric	$\hat{p}$=0.0395	-	126	3	0.8	0.10
Poisson	$\hat{p}$=25.33	-	412.6	4347625	0	0.48
logarithmic	$\hat{p}$=0.0001	-	$\to \infty$	$\to \infty$	0	0.83
Erlang-geometric	$\hat{p}$=0.1	$\hat{\lambda}$=0.395	126	3	0.8	0.074
Erlang-Poisson	$\hat{p}$=2.005	$\hat{\lambda}$=0.079	132.49	8.135	0.228	0.2
Erlang-logarithmic	$\hat{p}$=0.0001	$\hat{\lambda}$=0.09	140.64	10.78	0.1	0.2

7 Conclusion

This research has studied discrete Erlang mixtures using the geometric, Poisson and logarithmic mixing distributions. The posterior distribution of the Erlang-geometric distribution was demonstrated as the Poisson. The Erlang-Poisson mixture and its posterior distribution were expressed in terms of the Modified Bessel function of the first kind. The posterior distribution of the Erlang-logarithmic distribution was shown to be the truncated Poisson distribution, and the posterior moments were expressed as Touchard polynomials. The moments of the mixed distributions were expressed in terms of moments of the mixing distributions. Additionally, the cumulant generating functions of the Erlang mixtures have been obtained from their moment generating functions. Moments of the Erlang mixtures have also been obtained from their cumulant generating functions as cumulants. Further, Bayesian method was applied in parameter estimation, where the posterior means are the Bayes estimators of the conditional (Erlang) distribution's parameter, assuming squared loss function. Method of moments and maximum likelihood estimates for the mixed distributions were obtained, and the goodness of fit test for the mixtures was performed by fitting them to two real data sets. The mixed distributions were seen to have better fits compared to their corresponding mixing distributions, and the Erlang-geometric distribution had the best fit among the three discrete Erlang mixtures.

Construction of discrete Erlang mixed distributions using more mixing distributions, and further applications of the mixed distributions, are recommendations for further research.

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Competing Interests

Author has declared that no competing interests exist.

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