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Fuzzy d-algebras under t-norms

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Abstract: In this paper, by using t-norms, we introduce fuzzy subalgebras and fuzzy d-ideals of d-algebra and investigate some properties of them. Moreover, we define the cartesian product and intersection of fuzzy subalgebras and fuzzy d-ideals of d-algebra. Finally, by homomorphisms of d-algebras, we consider the image and pre-image of them.

Keywords: Algebra and orders; Theory of fuzzy sets; Norms; Products and intersections; Homomorphisms.

1. Introduction and Preliminaries

Neggers and Kim [1] introduced the notion of d-algebras and investigated the properties of them. Neggers *et al.* [2] introduced the concepts of d-ideals in d-algebra. Urge to deal with uncertainty by tools different from that of probability lead the way to fuzzy sets, rough sets and soft sets. Zadeh introduced fuzzy sets [3]. Akram and Dar [4] introduced the notions of fuzzy subalgebras and d-ideals in d-algebras and investigated some of their results. Al-Shehrie [5] introduced the notions of fuzzy dot d-ideals of d-algebras and some properties are investigated. Dejen [6] investigated product of fuzzy dot d-ideals and strong fuzzy relation and the corresponding strong fuzzy dot d-ideals. The triangular norms, t-norms, originated from the studies of probabilistic metric spaces in which triangular inequalities were extended using the theory of T-norms. Later, Hohle [7], Alsina *et al.* [8] introduced the t-norms into fuzzy set theory and suggested that the t-norms be used for the intersection of fuzzy sets. Since then, many other researchers presented various types of t-norms for particular purposes [9,10].

The author by using norms, investigated some properties of fuzzy algebraic structures [11–15]. In this paper, we introduce the notion of fuzzy subalgebras (as $FST(X)$) and fuzzy d-ideals (as $FDIT(X)$) of d-algebras X by using t-norm T and then we investigate different characterizations and several basic properties which are related to them. Next we define cartesian product and intersection of them and we obtain some new results about them. Finally we show that the image and pre-image of them are also $FST(X)$ and $FDIT(X)$ uner homomorphisms of d-algebras.

2. Preliminaries

The following definitions and preliminaries are required in the sequel of our work and hence presented in brief.

Definition 1. [1] A nonempty set X with a constant 0 and a binary operation $*$ is called a d-algebra, if it satisfies the following axioms:

1. $x * x = 0$,
2. $0 * x = 0$,
3. if $x * y = 0$ and $y * x = 0$, then $x = y$, for all $x, y \in X$.

Definition 2. [2] Let S be a non-empty subset of a d-algebra X, then S is called subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Definition 3. [2] Let X be a d-algebra and I be a subset of X, then I is called d-ideal of X if it satisfies following conditions:

1. $0 \in I$,
2. if $x * y \in I$ and $y \in I$, then $x \in I$,
3. if $x \in I$ and $y \in X$, then $x * y \in I$.

Definition 4. [1] A mapping $f : X \to Y$ of d-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$.

Definition 5. [16] Let X be an arbitrary set. A fuzzy subset of X, we mean a function from X into $[0,1]$. The set of all fuzzy subsets of X is called the $[0,1]$-power set of X and is denoted $[0,1]^X$. For a fixed $s \in [0,1]$, the set $\mu_s = \{x \in X : \mu(x) \geq s\}$ is called an upper level of μ.

Definition 6. [16] Let $f : X \to Y$ be a mapping of sets and $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. Define $f(\mu) \in [0,1]^Y$ and $f^{-1}(\nu) \in [0,1]^X$, defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(x) \mid x \in G, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

for all $y \in Y$. Also $f^{-1}(\nu)(x) = \nu(f(x))$ for all $x \in X$.

Definition 7. [17] A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

1. $T(x,1) = x$ (neutral element),
2. $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
3. $T(x,y) = T(y,x)$ (commutativity),
4. $T(x, T(y,z)) = T(T(x,y),z)$ (associativity),

for all $x, y, z \in [0,1]$.

We say that T is idempotent if for all $x \in [0,1]$,$T(x,x) = x$.

Example 1. The basic t-norms are $T_m(x,y) = min\{x,y\}$,$T_b(x,y) = max\{0, x+y-1\}$ and $T_p(x,y) = xy$, which are called standard intersection, bounded sum and algebraic product respectively.

Lemma 1. [17] *Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

3. Main results

Definition 8. Let μ be a fuzzy subset in d-algebra X. Then μ is called a fuzzy subalgebra of X under t-norm T iff $\mu(x * y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$. Denote by $FST(X)$, the set of all fuzzy subalgebras of X under t-norm T.

Example 2. Let $X = \{0,1,2\}$ be a set given by the following Cayley table:

*	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Then $(X, *, 0)$ is a d-algebra. Define fuzzy subset $\mu : (X, *, 0) \to [0,1]$ as

$$\mu(x) = \begin{cases} 0.35 & \text{if } x = 0, \\ 0.25 & \text{if } x \neq 0. \end{cases}$$

$T(a,b) = T_p(a,b) = ab$ for all a,b $\in [0,1]$ then $\mu \in FST(X)$.

In the following propositions we investigate relation between $\mu \in FST(X)$ and subalgebras of X.

Proposition 1. *Let $\mu \in [0,1]^X$ and T be idempotent. Then $\mu \in FST(X)$ if and only if the upper level μ_t is either empty or a subalgebra of X for every $t \in [0,1]$.*

Proof. Let $\mu \in FST(X)$ and $x, y \in \mu_t$. Then

$$\mu(x * y) \geq T(\mu(x), \mu(y)) \geq T(t,t) = t.$$

Thus $x * y \in \mu_t$ and so μ_t will be a subalgebra of X for every $t \in [0,1]$.

Conversely, let μ_t is either empty or a subalgebra of X for every $t \in [0,1]$. Let $t = T(\mu(x), \mu(y))$ and $x, y \in S$. As μ_t is a subalgebra of X so $x * y \in \mu_t$ and thus $\mu(x * y) \geq t = T(\mu(x), \mu(y))$. Then $\mu \in FST(X)$. □

In the following proposition we prove that any subalgebra of a d-algebra X can be realized as a level subalgebra of some fuzzy subalgebra of X.

Proposition 2. *Let A be a subalgebra of a d-algebra X and $\mu \in [0,1]^S$ such that*

$$\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

with $t \in (0,1)$. If T be idempotent, then $\mu \in FST(X)$.

Proof. We know that $A = \mu_t$. Let $x, y \in X$ and we investigate the following conditions;

1. If $x, y \in A$, then $x * y \in A$ and so
$$\mu(x * y) = t \geq t = T(t,t) = T(\mu(x), \mu(x)).$$
2. If $x \in A$ and $y \notin A$, then $\mu(x) = t$ and $\mu(y) = 0$ and so
$$\mu(x * y) \geq 0 = T(t,0) = T(\mu(x), \mu(y)).$$
3. If $x \notin A$ and $y \in A$, then $\mu(x) = 0$ and $\mu(y) = t$ and so
$$\mu(x * y) \geq 0 = T(0,t) = T(\mu(x), \mu(y)).$$
4. If $x \notin A$ and $y \notin A$, then $\mu(x) = 0$ and $\mu(y) = 0$ and so
$$\mu(x * y) \geq 0 = T(0,0) = T(\mu(x), \mu(y)).$$

Thus from (1)-(4) we get that $\mu \in FST(X)$.

□

Corollary 1. *Let A be a subset of X. Then the characteristic function $\chi_A \in FST(X)$ if and only if A is a subalgebra of X.*

Now under some conditions we prove that $\mu_s = \mu_t$ for every $s, t \in [0,1]$.

Proposition 3. *Let $\mu \in FST(X)$ and $s, t \in [0,1]$. If $s < t$, then $\mu_s = \mu_t$ if and only if there is no $x \in X$ such that $s \leq \mu(x) < t$.*

Proof. Let $s < t$ and $\mu_s = \mu_t$. If there exists $x \in X$ such that $s \leq \mu(x) < t$, then $x \in \mu_s$ but $x \notin \mu_t$ which is contradicting the hypothesis.

Conversely, let there is no $x \in X$ such that $s \leq \mu(x) < t$. As $x \in \mu_s$ so $x \in \mu_t$ then $\mu_s \subseteq \mu_t$. If $x \in \mu_t$ then $\mu(x) \geq t > s$ so $x \in \mu_s$ then $\mu_t \subseteq \mu_s$. Therefore $\mu_s = \mu_t$. □

Definition 9. $\mu \in [0,1]^X$ is called fuzzy d-ideal of X under t-norm T if it satisfies the following inequalities:

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq T(\mu(x * y), \mu(y))$,
3. $\mu(x * y) \geq T(\mu(x), \mu(y))$, for all $x, y \in X$.

The set of all fuzzy d-ideals of X under t-norm T is denoted by $FDIT(X)$.

Corollary 2. *Let $\mu \in FDIT(X)$. Then*

1. $\mu \in FST(X)$.
2. $\mu(0) \geq \mu(x)$ *and* $\mu(x) \geq T(T(\mu(x), \mu(y)), \mu(y))$ *for all* $x, y \in X$.

Example 3. Let $X = \{0, 1, 2, 3\}$ be a set given by the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	2	0	0
3	3	3	3	0

Then $(X, *, 0)$ is a d-algebra. Define fuzzy subset $\mu : (X, *, 0) \rightarrow [0,1]$ as

$$\mu(x) = \begin{cases} 0.65 & \text{if } x = 0 \\ 0.15 & \text{if } x \neq 0 \end{cases}$$

$T(a,b) = T_p(a,b) = ab$ for all a,b $\in [0,1]$ then $\mu \in FDIT(X)$.

Definition 10. Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. The cartesian product of μ and ν is denoted by $\mu \times \nu : X \times Y \rightarrow [0,1]$ and is defined by $(\mu \times \nu)(x,y) = T(\mu(x), \nu(y))$ for all $(x,y) \in X \times Y$.

In the following propositions we investigate the properties of cartesian product $FST(X)$ and $FDIT(X)$.

Proposition 4. *Let $\mu \in FST(X)$ and $\nu \in FST(Y)$. Then $\mu \times \nu \in FST(X \times Y)$.*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2) \\
&= T(\mu(x_1 * x_2), \nu(y_1 * y_2)) \\
&\geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2))) \\
&= T(T(\mu(x_1), \nu(y_1)), T(\mu(x_2), \nu(y_2))) \text{ (Lemma 1)} \\
&= T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)).
\end{aligned}$$

Thus

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) \geq T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2))$$

and so $\mu \times \nu \in FST(X \times Y)$. □

Proposition 5. *Let $\mu \in FDIT(X)$ and $\nu \in FDIT(X)$. Then $\mu \times \nu \in FDIT(X \times Y)$.*

Proof. 1. Let $(x,y) \in X \times Y$. Then $(\mu \times \nu)(0,0) = T(\mu(0),)\nu(0)) \geq T(\mu(x),)\nu(x))$.

2. Let $x_i \in X$ and $y_i \in Y$ for $i = 1, 2$, then

$$\begin{aligned}(\mu \times \nu)(x_1, y_1) &= T(\mu(x_1), \nu(y_1)) \\ &\geq T(T(\mu(x_1 * x_2), \mu(x_2)), T(\nu(y_1 * y_2), \nu(y_2))) \\ &= T(T(\mu(x_1 * x_2), \nu(y_1 * y_2)), T(\mu(x_2), \nu(y_2))) \text{ (Lemma 1)} \\ &= T((\mu \times \nu)(x_1 * x_2, y_1 * y_2), (\mu \times \nu)(x_2, y_2)) \\ &= T((\mu \times \nu)(x_1, y_1) * (x_2, y_2), (\mu \times \nu)(x_2, y_2)).\end{aligned}$$

Then $(\mu \times \nu)(x_1, y_1) \geq T((\mu \times \nu)(x_1, y_1) * (x_2, y_2), (\mu \times \nu)(x_2, y_2))$.

3.

$$\begin{aligned}(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2) \\ &= T(\mu(x_1 * x_2), \nu(y_1 * y_2)) \\ &\geq T(T(\mu(x_1), \mu(x_2)), T(\nu(y_1), \nu(y_2))) \\ &= T(T(\mu(x_1), \nu(y_1)), T(\mu(x_2), \nu(y_2))) \text{ (Lemma 1)} \\ &= T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)).\end{aligned}$$

Thus

$$(\mu \times \nu)((x_1, y_1) * (x_2, y_2)) \geq T((\mu \times \nu)(x_1, y_1), (\mu \times \nu)(x_2, y_2)).$$

Therefore from (1) -(3) we get that $\mu \times \nu \in FDIT(X \times Y)$. □

Proposition 6. *Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. If $\mu \times \nu \in FDIT(X \times Y)$, then at least one of the following two statements must hold.*

1. *$\mu(0) \geq \mu(x)$ for all $x \in X$.*
2. *$\nu(0) \geq \nu(y)$ for all $y \in Y$.*

Proof. Let none of the statements (1) and (2) holds, then we can find $(x, y) \in X \times Y$ such that $\mu(0) < \mu(x)$ and $\nu(0) < \nu(y)$. Thus

$$(\mu \times \nu)(x, y) = T(\mu(x), \nu(y)) > T(\mu(0), \nu(0)) = (\mu \times \nu)(0, 0)$$

and it is contradiction with $\mu \times \nu \in FDIT(X \times Y)$. □

Proposition 7. *Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$. If $\mu \times \nu \in FDIT(X \times Y)$ and T be idempotent, then we obtain the following statements:*

1. *If $\mu(0) \geq \mu(x)$, then either $\nu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(y)$ for all $(x, y) \in X \times Y$.*
2. *If $\nu(0) \geq \nu(y)$, then either $\mu(0) \geq \nu(y)$ or $\mu(0) \geq \mu(x)$ for all $(x, y) \in X \times Y$.*

Proof. 1. Let $\mu(0) \geq \mu(x)$ and we have $(x, y) \in X \times Y$ such that $\nu(0) < \mu(x)$ and $\nu(0) < \nu(y)$. Then $\mu(0) \geq \mu(x) > \nu(0)$ and so $\nu(0) = T(\mu(0), \nu(0))$. Thus

$$(\mu \times \nu)(x, y) = T(\mu(x), \nu(y)) > T(\nu(0), \nu(0)) = \nu(0) = T(\mu(0), \nu(0)) = (\mu \times \nu)(0, 0)$$

and it is contradiction with $\mu \times \nu \in FDIT(X \times Y)$.

2. Let $\nu(0) \geq \nu(y)$ such that for $(x, y) \in X \times Y$ we have $\mu(0) < \nu(y)$ and $\mu(0) < \mu(x)$. So $\nu(0) \geq \nu(y) > \mu(0)$ and $\mu(0) = T(\mu(0), \nu(0))$. Thus

$$(\mu \times \nu)(x, y) = T(\mu(x), \nu(y)) > T(\mu(0), \mu(0)) = \mu(0) = T(\mu(0), \nu(0)) = (\mu \times \nu)(0, 0)$$

and it is contradiction with $\mu \times \nu \in FDIT(X \times Y)$. □

Now we prove the converse of Proposition 5.

Proposition 8. *If $\mu \times \nu \in FDIT(X \times Y)$ and T be idempotent, then $\mu \in FDIT(X)$ or $\nu \in FDIT(Y)$.*

Proof. We prove that $\mu \in FDIT(X)$ and the proof $\nu \in FDIT(Y)$ is similar. As Proposition 6 we have that

$$\mu(0) \geq \mu(x) \tag{1}$$

for all $x \in X$ and from Proposition 7(1), we have either $\nu(0) \geq \mu(x)$ or $\nu(0) \geq \nu(y)$ for all $(x,y) \in X \times Y$ thus $(\mu \times \nu)(x,0) = T(\mu(x), \nu(0)) = \mu(x)$. Let $(x,y), (\acute{x}, \acute{y}) \in X \times Y$ and as $\mu \times \nu \in FDIT(X \times Y)$ so

$$(\mu \times \nu)(x,y) \geq T((\mu \times \nu)((x,y) * (\acute{x}, \acute{y})), (\mu \times \nu)(\acute{x}, \acute{y}))) = T((\mu \times \nu)((x * \acute{x}, y * \acute{y})), (\mu \times \nu)(\acute{x}, \acute{y})))$$

thus

$$(\mu \times \nu)(x,y) \geq T((\mu \times \nu)((x * \acute{x}, y * \acute{y})), (\mu \times \nu)(\acute{x}, \acute{y})))$$

and by putting $y = \acute{y} = 0$ we will have

$$(\mu \times \nu)(x,0) \geq T((\mu \times \nu)((x * \acute{x}, 0 * 0)), (\mu \times \nu)(\acute{x}, 0)))$$

and so

$$\mu(x) \geq T(\mu(x * \acute{x}), \mu(\acute{x})). \tag{2}$$

Also since $\mu \times \nu \in FDIT(X \times Y)$ so

$$(\mu \times \nu)((x,y) * (\acute{x}, \acute{y})) \geq T((\mu \times \nu)(x,y), (\mu \times \nu)(\acute{x}, \acute{y}))$$

thus

$$(\mu \times \nu)(x * \acute{x}, y * \acute{y}) \geq T((\mu \times \nu)(x,y), (\mu \times \nu)(\acute{x}, \acute{y}))$$

and by letting $y = \acute{y} = 0$ we get that

$$(\mu \times \nu)(x * \acute{x}, 0 * 0) \geq T((\mu \times \nu)(x,0), (\mu \times \nu)(\acute{x}, 0))$$

which means that

$$\mu(x * \acute{x}) \geq T(\mu(x), \mu(\acute{x})). \tag{3}$$

Thus from Eqs. (1)-(3), we have that $\mu \in FDIT(X)$. $\square$

Definition 11. Let $A : S \to [0,1]$ be a fuzzy set in a set S. The strongest fuzzy relation on S under t-norm T is fuzzy relation on A with $\mu_A : S \times S \to [0,1]$ given by

$$\mu_A(x,y) = T(A(x), A(y))$$

for all $x, y \in S$.

Proposition 9. *Let T be idempotent. Then*

$$A \in FDIT(X) \Longleftrightarrow \mu_A \in FDIT(X \times X).$$

Proof. Let $A \in FDIT(X)$.

1. Let $x \in X$ then $\mu_A(0,0) = T(A(0), A(0)) \geq T(A(x), A(x)) = \mu_A(x,x)$.
2. Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}
\mu_A(x_1, x_2) &= T(A(x_1), A(x_2)) \\
&\geq T(T(A(x_1 * y_1), A(y_1)), T(A(x_2 * y_2), A(y_2))) \\
&= T(T(A(x_1 * y_1), A(x_2 * y_2)), T(A(y_1), A(y_2))) && \text{(Lemma 1)} \\
&= T(\mu_A(x_1 * y_1, x_2 * y_2), \mu_A(y_1, y_2)) \\
&= T(\mu_A((x_1, x_2) * (y_1, y_2)), \mu_A(y_1, y_2)).
\end{aligned}$$

Thus
$$\mu_A(x_1,x_2) \geq T(\mu_A((x_1,x_2)*(y_1,y_2)),\mu_A(y_1,y_2)).$$

3. Let $(x_1,x_2),(y_1,y_2) \in X \times X$. Then
$$\begin{aligned}\mu_A((x_1,x_2)*(y_1,y_2)) &= \mu_A(x_1*y_1,x_2*y_2)\\ &= T(A(x_1*y_1),A(x_2*y_2))\\ &\geq T(T(A(x_1),A(y_1)),T(A(x_2),A(y_2))) \qquad \text{(Lemma 1)}\\ &= T(T(A(x_1),A(x_2)),T(A(y_1),A(y_2)))\\ &= T(\mu_A(x_1,x_2),\mu_A(y_1,y_2)).\end{aligned}$$
So
$$\mu_A((x_1,x_2)*(y_1,y_2)) \geq T(\mu_A(x_1,x_2),\mu_A(y_1,y_2)).$$

Then (1)-(3) give us $\mu_A \in FDIT(X \times X)$.

Conversely, suppose that $\mu_A \in FDIT(X \times X)$.

1. Let $x \in X$ then
$$A(0) = T(A(0),A(0)) = \mu_A(0,0) \geq \mu_A(x,x) = A(x)$$
and
$$A(0) \geq A(x).$$
So $\mu_A(x,0) = T(A(x),A(0)) = A(x)$.
2. Let $(x_1,x_2),(y_1,y_2) \in X \times X$, then
$$\mu_A(x_1,x_2) \geq T(\mu_A(x_1,x_2)*(y_1,y_2)),\mu_A(y_1,y_2))) = T(\mu_A)(x_1)*y_1,(x_2*(y_2),\mu_A(y_1,y_2)).$$
If we let $x_2 = y_2 = 0$, then
$$\mu_A(x_1,0) \geq T(\mu_A(x_1*y_1,0*0),\mu_A(y_1,0)).$$
Thus $A(x_1) \geq T(A(x_1*y_1),A(y_1))$.
3. Let $(x_1,x_2),(y_1,y_2) \in X \times X$, then
$$\mu_A((x_1,x_2)*(y_1,y_2)) \geq T(\mu_A(x_1,x_2),\mu_A(y_1,y_2))$$
and
$$\mu_A(x_1*y_1,x_2*y_2) \geq T(\mu_A(x_1,x_2),\mu_A(y_1,y_2)).$$
By letting $x_2 = y_2 = 0$, we get that
$$\mu_A(x_1*y_1,0)*0) \geq T(\mu_A(x_1,0),\mu_A(y_1,0))$$
and thus $A(x_1*y_1) \geq T(A(x_1),A(y_1))$.

Now, from (1)-(3), we have $A \in FDIT(X)$. □

Definition 12. Let $\mu \in [0,1]^X$ and $\nu \in [0,1]^X$. The intersection of μ and ν is denoted by $\mu \cap \nu : X \to [0,1]$ and is defined by $(\mu \cap \nu)(x) = T(\mu(x),\nu(x))$ for all $x \in X$.

In the following propositions we investigate the intersection of two $\mu,\nu \in FST(X)$ and $\mu,\nu \in FDIT(X)$.

Proposition 10. *If $\mu,\nu \in FST(X)$, then $\mu \cap \nu \in FST(X)$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}(\mu \cap \nu)(x * y) &= T(\mu(x * y), \nu(x * y)) \\ &\geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).\end{aligned}$$

Thus $(\mu \cap \nu)(x * y) \geq T((\mu \cap \nu)(x), (\mu \cap \nu)(y))$ and so $\mu \cap \nu \in FST(X)$. □

Proposition 11. *If $\mu, \nu \in FDIT(X)$, then $\mu \cap \nu \in FDIT(X)$.*

Proof. Let $x, y \in X$. Then

1. $$(\mu \cap \nu)(0) = T(\mu(0), \nu(0)) \geq T(\mu(x), \nu(x)) = (\mu \cap \nu)(x).$$
2. $$\begin{aligned}(\mu \cap \nu)(x) &= T(\mu(x), \nu(x)) \\ &\geq T(T(\mu(x * y), \mu(y)), T(\nu(x * y), \nu(y))) \\ &= T(T(\mu(x * y), \nu(x * y)), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(x * y), (\mu \cap \nu)(y))\end{aligned}$$
   and thus
   $$(\mu \cap \nu)(x) \geq T((\mu \cap \nu)(x * y), (\mu \cap \nu)(y)).$$
3. $$\begin{aligned}(\mu \cap \nu)(x * y) &= T(\mu(x * y), \nu(x * y)) \\ &\geq T(T(\mu(x), \mu(y)), T(\nu(x), \nu(y))) \\ &= T(T(\mu(x), \nu(x)), T(\mu(y), \nu(y))) \\ &= T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).\end{aligned}$$
   Then
   $$(\mu \cap \nu)(x * y) \geq T((\mu \cap \nu)(x), (\mu \cap \nu)(y)).$$

Now from (1)-(3), we get $\mu \cap \nu \in FDIT(X)$. □

In the following propositions we consider $FST(X)$ and $FDIT(X)$ under homomorphisms of d-algebras.

Proposition 12. *If $\mu \in FST(X)$ and $f : X \to Y$ be a homomorphism of d-algebras, then $f(\mu) \in FST(Y)$.*

Proof. Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned}f(\mu)(y_1 * y_2) &= \sup\{\mu(x_1 * x_2) \mid x_1, x_2 \in X, f(x_1) = y_1, f(x_2) = y_2\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1, x_2 \in X, f(x_1) = y_1, f(x_2) = y_2\} \\ &= T(\sup\{\mu(x_1) \mid x_1 \in X, f(x_1) = y_1\}, \sup\{\mu(x_2) \mid x_2 \in X, f(x_2) = y_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)).\end{aligned}$$

Thus

$$f(\mu)(y_1 * y_2) \geq T(f(\mu)(y_1), f(\mu)(y_2))$$

and then $f(\mu) \in FST(Y)$. □

Proposition 13. *If $\nu \in FST(Y)$ and $f : X \to Y$ be a homomorphism of d-algebras, then $f^{-1}(\nu) \in FST(X)$.*

Proof. Let $x_1, x_2 \in X$. Then

$$\begin{aligned} f^{-1}(\nu)(x_1 * x_2) &= \nu(f(x_1 * x_2)) \\ &= \nu(f(x_1) * f(x_2)) \\ &\geq T(\nu(f(x_1)), \nu(f(x_2))) \\ &= T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)). \end{aligned}$$

Thus

$$f^{-1}(\nu)(x_1 * x_2) \geq T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2))$$

then $f^{-1}(\nu) \in FST(X)$. □

Proposition 14. *If $\mu \in FDIT(X)$ and $f : X \to Y$ be a homomorphism of d-algebras, then $f(\mu) \in FDIT(Y)$.*

Proof. 1. Let $x \in X$ and $y \in Y$ with $f(x) = y$. Now

$$f(\mu)(0) = \sup\{\mu(0) \mid 0 \in X, f(0) = 0\} \geq \sup\{\mu(x) \mid x \in X, f(x) = y\} = f(\mu)(y).$$

2. Let $x, x_1 \in X$ such that $f(x) = y, f(x_1) = y_1$. Now

$$\begin{aligned} f(\mu)(y) &= \sup\{\mu(x) \mid x \in X, f(x) = y\} \\ &\geq \sup\{T(\mu(x * x_1), \mu(x_1)) \mid x, x_1 \in X, f(x) = y, f(x_1) = y_1\} \\ &= T(\sup\{\mu(x * x_1) \mid x, x_1 \in X, f(x) = y, f(x_1) = y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, f(x_1) = y_1\}) \\ &= T(\sup\{\mu(x * x_1) \mid x, x_1 \in X, f(x * x_1) = y * y_1\}, \sup\{\mu(x_1) \mid x_1 \in X, f(x_1) = y_1\} \\ &= T(f(\mu)(y * y_1), f(\mu)(y_1)). \end{aligned}$$

Therefore

$$f(\mu)(y) \geq T(f(\mu)(y * y_1), f(\mu)(y_1)).$$

3. Let $y_1, y_2 \in Y$ and $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then

$$\begin{aligned} f(\mu)(y_1 * y_2) &= \sup\{\mu(x_1 * x_2) \mid x_1, x_2 \in X, f(x_1) = y_1, f(x_2) = y_2\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2) \mid x_1, x_2 \in X, f(x_1) = y_1, f(x_2) = y_2\} \\ &= T(\sup\{\mu(x_1) \mid x_1 \in X, f(x_1) = y_1\}, \sup\{\mu(x_2) \mid x_2 \in X, f(x_2) = y_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

Thus from (1)-(3), we have that $f(\mu) \in FDIT(Y)$. □

Proposition 15. *If $\nu \in FDIT(Y)$ and $f : X \to Y$ be a homomorphism of d-algebras, then $f^{-1}(\nu) \in FDIT(X)$.*

Proof. 1. Let $x \in X$. Then

$$f^{-1}(\nu)(0) = \nu(f(0)) \geq \nu(f(x) = f^{-1}(\nu)(x).$$

2. Let $x, x_1 \in X$. As

$$\begin{aligned} f^{-1}(\nu)(x) &= \nu(f(x)) \\ &\geq T(\nu(f(x) * f(x_1)), \nu(f(x))) \\ &= T(\nu(f(x * x_1)), \nu(f(x))) \\ &= T(f^{-1}(\nu)(x * x_1), f^{-1}(\nu)(x)). \end{aligned}$$

So

$$f^{-1}(\nu)(x) \geq T(f^{-1}(\nu)(x * x_1), f^{-1}(\nu)(x)).$$

3. Let $x_1, x_2 \in X$. Then

$$\begin{aligned} f^{-1}(\nu)(x_1 * x_2) &= \nu(f(x_1 * x_2)) \\ &= \nu(f(x_1) * f(x_2)) \\ &\geq T(\nu(f(x_1)), \nu(f(x_2))) \\ &= T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)). \end{aligned}$$

Then

$$f^{-1}(\nu)(x_1 * x_2) \geq T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)).$$

Therefore, from (1)-(3,) we have $f^{-1}(\nu) \in FDIT(X)$. $\square$

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