A New Generalized Weibull- Odd Frèchet Family of Distributions: Statistical Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. Author AU designed the study, performed the analysis and wrote the first draft of the manuscript. Author SISD modified the entire work with good suggestions and performing some derivations while authors BBA and ATI check over and corrected all the derivations. All authors read and approved the final manuscript.

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Abstract

We introduced a new generalized Weibull- Odd Frèchet family of distributions with three extra parameters and we derived some of its structural properties. We derived comprehensive mathematical properties which include moments, moment generating function, Entropies and Order Statistics. One family of this distribution called new generalized Weibull- Odd Frèchet -Frèchet distribution is used to fit two data sets using the MLE procedure. A Monte Carlo simulation is used to test the robustness of the parameters of this distribution, in terms of the bias and mean squared error. The results of fitting this new distribution to two different data sets suggest that the new distribution outperforms its competitors.

1 Introduction

Statistical distributions have received reasonable attention by those working in both theory and application because the Statistical analysis depends heavily on the statistical distribution to address any problem under

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study; everyday several models are proposed in order to make it more tractable and flexible. The tractability of a probability distribution makes it easier for the researcher; especially when it comes to simulation of random samples, but the flexibility of probability distributions is of interest because more flexible models give more information than the less flexible models. It is advisable to use the probability distributions that best fit the dataset than to alter the already existing distribution as this may affect the originality of the dataset. As a result of this, several efforts have been made in recent years to ensure that the existing standard distributions are modified; this includes Transmuted Rayleigh distribution by Merovci F. [1].

Exponential distribution was first generalized by Gupta and Kundu [2] and named it as Exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Some others generators are beta-G by Eugene et al. [3], Kumaraswamy family by Cordeiro and de Castro [4], Exponentiated generalized class of distributions by Cordeiro et al. [5], new technique for Generating Families of probability distribution function as a generator by Alzaatreh et al. [6], the Lomax Generator of distributions by Cordeiro et al. [7], beta Marshall-Olkin family of distributions by Alizadeh et al. [8], Kumaraswamy Marshall-Olkin family of distributions by Alizadeh et al. [9], Kumaraswamy transmuted-G family of distributions by Afify et al. [10] and many more.

1.1 Weibull distribution

Weibull distribution by Weibull [11] is one of the most widely used lifetime distribution and has been identified as a life testing model in reliability and engineering. It is a distribution that can take a form of other types of distributions, based on the value of the shape parameter β. For example, the Weibull distribution reduces to exponential distribution when the shape parameter $\beta=1$. In reliability analysis, the Weibull distribution can be used to find the percentage of items that are expected to fail during the burn-in period.

A random variable X is said to follow a Weibull distribution, if its CDF and pdf are respectively given by Equation (1) and (2)

$$F(x;\alpha,\beta) = 1 - exp\left[-\alpha x^{\beta}\right], x \geq 0, \alpha > 0, \beta > 0 \tag{1}$$

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1} exp\left[-\alpha x^{\beta}\right], x \geq 0, \alpha > 0, \beta > 0 \tag{2}$$

Where α and β are scale and shape parameters respectively.

1.2 Frèchet distribution

French mathematician introduced Frèchet distribution by Maurice Frèchet [12]. This Frèchet distribution has been described as a distribution for modeling and analysis of several extreme events ranging from accelerated life testing to earthquakes, floods, rain fall, sea currents and wind speeds. Applications of the Frèchet distribution in various fields given in Harlow [13] showed that it is an important distribution for modeling the statistical behavior of materials properties for a variety of engineering applications. Frèchet distribution can also be called the Inverse Weibull distribution

A random variable X is said to follow a Frèchet distribution with one parameter, if its cumulative distribution function (cdf) and probability density function (pdf) are respectively given by equation (3) and (4)

$$F(x;\lambda) = exp\left[-\left(\frac{1}{x}\right)^{\lambda}\right], \qquad x \geq 0, \lambda > 0 \tag{3}$$

And

$$f(x;\lambda) = \frac{\lambda}{x^{\lambda+1}} exp\left[-\left(\frac{1}{x}\right)^{\lambda}\right]; \quad x \geq 0, \lambda > 0 \tag{4}$$

Where λ is the shape parameter.

This article is structured as follows. Section 2, presents the proposed new Generalized Weibull odd Frèchet (NGWoFr) family of distributions and one of its sub-model. The structural properties of the proposed family including some useful expansions are presented in section 3, section 4 contains some parameter estimation technique and some simulation study in order to assess the robustness of the estimated parameter. Data analysis in the form of fitting the new Generalized Weibull odd Frèchet –Frechet distribution and its competitors to the life testing and length of intervals between the times at which vehicles pass a point on a road is presented in section 5 while section 6 concludes the paper.

2 The New Generalized Weibull- Odd Frèchet (NGWoFr) Family

This section introduced the new proposed family of distributions.

2.1 The new generalized Weibull family of distribution

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative functions of the baseline model with parameter vector ξ and consider the Weibull distribution cdf $F(t) = 1 - \exp(-\alpha t^{\beta})$ (for $t > 0$) with positive parameters α and β. Then Cordeiro et al. [14] defined the cdf of the Weibull-G Family of Distributions by replacing the argument of t by $-\log[1-G(x;\xi)]$ as follow

$$F(x;\alpha,\beta,\xi) = \alpha\beta\int_0^{-\log[1-G(x;\xi)]} t^{\beta-1} \exp(-\alpha t^{\beta})dt = 1 - exp\left[-\alpha\left(-\log[1-G(x;\xi)]\right)^{\beta}\right], \alpha,\beta > 0 \tag{5}$$

The corresponding pdf is given by

$$f(x;\alpha,\beta,\xi) = \frac{\alpha\beta g(x;\xi)}{[1-G(x;\xi)]}\left\{-\log[1-G(x;\xi)]\right\}^{\beta-1} exp\left[-\alpha\left(-\log[1-G(x;\xi)]\right)^{\beta}\right] \tag{6}$$

2.2 The odd Frèchet-G family of probability distributions

Let $g(x;\xi)$ and $G(x;\xi)$ denote the density and cumulative distribution functions of the baseline model with parameter vector ξ and consider the Frèchet cdf $F(t,\lambda) = exp\left[-\left(\frac{1}{t}\right)^{\lambda}\right]$ (for $t \geq 0$) with positive parameter $\lambda > 0$, then the cdf of the Odd Frèchet-G family proposed by Ulhaq et al.[15] is defined by replacing the argument of t by $G(x;\xi)/\bar{G}(x;\xi)$, where $\bar{G}(x;\xi) = 1 - G(x;\xi)$ as follow

$$F_{OFG}(x;\lambda,\xi) = \int_0^{\left(\frac{G(x;\xi)}{\bar{G}(x;\xi)}\right)} \frac{1}{t^{\lambda+1}} exp\left[-\left(\frac{1}{t}\right)^{\lambda}\right]dx = \exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]; \qquad \lambda > 0 \qquad (7)$$

The corresponding pdf is given by

$$f_{OFG}(x;\lambda,\xi) = \frac{\lambda g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{\left[G(x;\xi)\right]^{\lambda+1}} \exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]; \qquad \lambda > 0 \qquad (8)$$

2.3 New generalized Weibull- odd Frèchet (NGWoFr) family of probability distributions

We defined the proposed family (New Generalized Weibull- odd Frèchet Family of Distributions) via the cdf and pdf

$$F(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\int_0^{H(x;\lambda,\xi)} t^{\beta-1}\exp(-\alpha t^{\beta})dt = 1-\exp\left\{-\alpha[\left(-\log\left(1-[F_{OFG}(t;\lambda,\xi)]\right)\right)]^{\beta}\right\} \quad (9)$$

The corresponding pdf is given by:

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \frac{\alpha\beta f_{OFG}(t;\lambda,\xi)}{[1-F_{OFG}(t;\lambda,\xi)]}\left(-\log[1-F_{OFG}(t;\lambda,\xi)]\right)^{\beta-1}\exp\left\{-\alpha\left(-\log[1-F_{OFG}(t;\lambda,\xi)]\right)^{\beta}\right\} \quad (10)$$

Where

$$H(t;\lambda,\xi) = \left(-\log\left(1-[F_{OFG}(t;\lambda,\xi)]\right)\right)$$

substitution equation (7) in equation (9) and (10), we have the cdf and corresponding pdf as:

$$F_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = 1-\exp\left\{-\alpha\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta}\right\} \quad (11)$$

And

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\lambda\frac{g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{\left[G(x;\xi)\right]^{\lambda+1}}\left[\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right]^{-1}$$

$$\times\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta-1}\exp\left\{-\alpha\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta}\right\} \quad (12)$$

2.3.1 Survival function

The survival function (R(x)) is the probability that a patient, device or any objects of interest will survive beyond a specified time, the survival function is also known as the reliability function. The survival function of New Generalized Weibull- odd Frèchet Family of Distributions is given by equation (13).

$$R(x) = 1 - F(x; \alpha, \beta, \lambda, \xi) = \exp\left\{ -\alpha \left(-\log\left(1 - \exp\left[-\left(\frac{1 - G(x; \xi)}{G(x; \xi)} \right)^{\lambda} \right] \right) \right)^{\beta} \right\}$$

(13)

2.3.2 Hazard function

Hazard rate, H(x) refers to the rate of death for an item of a given age, and it's also known as the failure rate. It analyses the likelihood that something will survive to an earlier time t. In other words, it is the likelihood that if something survives to one moment; it will also survive to the next. Hazard rate cannot be negative and only applies to those items which cannot be repaired. The hazard function of New Generalized Weibull- odd Frèchet Family of Distributions is given in equation (14).

$$H(x) = \frac{f(x; \alpha, \beta, \lambda, \xi)}{1 - F(x; \alpha, \beta, \lambda, \xi)} = \alpha\beta\lambda \frac{g(x; \xi)[1 - G(x; \xi)]^{\lambda-1}}{[G(x; \xi)]^{\lambda+1}} \left[\exp\left[\left(\frac{1 - G(x; \xi)}{G(x; \xi)} \right)^{\lambda} \right] - 1 \right]^{-1}$$

$$\times \left(-\log\left(1 - \exp\left[-\left(\frac{1 - G(x; \xi)}{G(x; \xi)} \right)^{\lambda} \right] \right) \right)^{\beta-1}$$

(14)

2.3.3 Quantile function

Quantile function also called inverse cumulative distribution function and is associated with probability distribution of a random variable used for simulation study and it's given by:

$$Q(u) = G(x; \xi)^{-1} \left(1 + \left(-\log\left(1 - \exp\left\{ -\left[-\frac{1}{\alpha} \log(1 - u) \right]^{\frac{1}{\beta}} \right\} \right)^{\frac{1}{\lambda}} \right) \right)^{-1}$$

(15)

Where $G(x; \xi)$ the cumulative distribution function of any continuous probability distribution and u is a random number generated from uniform distribution between 0 and 1.

In section (2.3.4), we provide sub-model of New Generalized Weibull- odd Frèchet Family of Distributions. The probability density function given in equation (11) will be most useful when the cdf $G(x; \xi)$ and the pdf $g(x; \xi)$ have simple analytic expressions.

2.3.4 New generalized Weibull- Odd Frèchet–Frèchet (NGWoFr-Fr) distribution

Suppose that the parent distribution is Frechetwith cdf and pdf respectively given as $G(x;\theta,\gamma) = exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]$ and $g(x;\theta,\gamma) = \theta^{\gamma}\gamma x^{-(\gamma+1)}exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]$ for (x> 0), then the New Generalized Weibull Odd Frèchet- Frèchet distribution has the cdf and corresponding pdf respectively given by equations (16) and (17)

$$F_{W-OFG}(x;\alpha,\beta,\lambda,\theta,\gamma) = 1 - exp\left\{-\alpha\left(-\log\left(1-exp\left[-\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)\right]^{\lambda}\right)\right)^{\beta}\right\}$$

(16)

And

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\theta,\gamma) = \alpha\beta\lambda\theta^{\gamma}\gamma \dfrac{x^{-(\gamma+1)}\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}}{\left(1-exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]\right)}\left[exp\left[\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}\right]-1\right]^{-1}$$

$$\times\left(-\log\left(1-exp\left[-\left(exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}\right]\right)\right)^{\beta-1}exp\left\{-\alpha\left(-\log\left(1-exp\left[-\left(exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}\right]\right)\right)^{\beta}\right\}$$

(17)

The survival R(x) and hazard h(x) function of New Generalized Weibull- odd Frèchet-Frèchet Family of Distributions are given in equation (18) and (19) respectively,

$$R(x) = 1 - F(x;\alpha,\beta,\theta,\lambda,\gamma) = exp\left\{-\alpha\left(-\log\left(1-exp\left[-\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)\right]^{\lambda}\right)\right)^{\beta}\right\}$$

(18)

and

$$(h(x)) = \alpha\beta\lambda\theta^{\gamma}\gamma\dfrac{x^{-(\gamma+1)}\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}}{\left(1-exp\left[-\left(\dfrac{\theta}{x}\right)^{\gamma}\right]\right)}\left[exp\left[\left(exp\left[\left(\dfrac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}\right]-1\right]^{-1}$$

(19)

$$\times \left(-\log\left(1-\exp\left[-\left(exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]-1\right)^{\lambda}\right]\right)\right)^{\beta-1}$$

(20)

The Quantile function of new Generalized Weibull Frèchet- Frèchet distribution is given by:

$$Q(u) = \theta\left(-\log\left(1+\left(-\log\left(1-\exp\left\{ -\left[-\frac{1}{\alpha}\log(1-u)\right]^{\frac{1}{\beta}}\right\}\right)^{\frac{1}{\lambda}}\right)^{-1}\right)\right)^{-\frac{1}{\gamma}}$$

(21)

$$0 < u < 1.$$

where u is a random number generated from uniform distribution between 0 and 1; that is

(a) Pdf of NGWFr-FrD   (b) hazard function of NGWFr-FrD
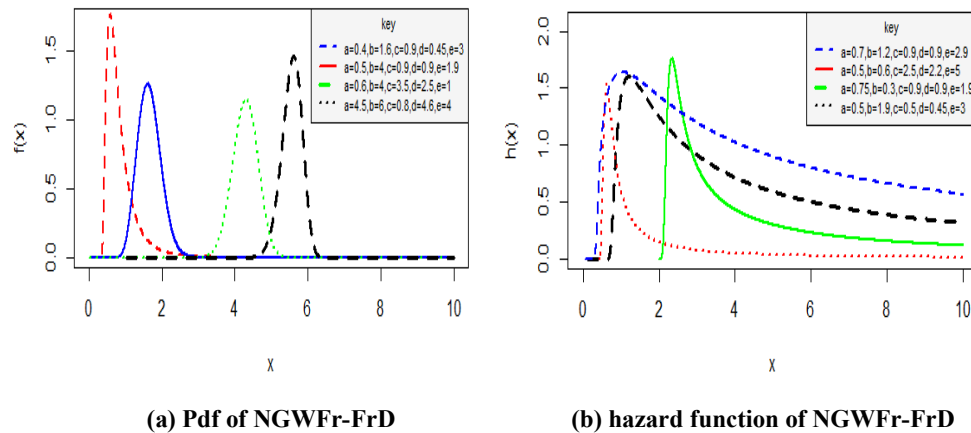
Fig. 1. Shows some possible shapes of probability density function and hazard function of new generalized Weibull- odd Frèchet-FrèchetDistribution for some selected values of the parameters, respectively

3 Some Structural Properties of the New Family of Distribution

This section provides some Statistical Properties of the new family of distributions.

3.1 Useful expansions of the proposed family of distributions

In this section, we provide a very useful expansion for the proposed New Generalized Weibull- odd Frèchet Family of Distributions.

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\lambda \frac{g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{[G(x;\xi)]^{\lambda+1}} \left[\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right]^{-1}$$

$$\times\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta-1} \exp\left\{-\alpha\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta}\right\} \quad (22)$$

Using the power series expansion

$$\exp\left\{-\alpha\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta}\right\} = \sum_{i=0}^{\infty}\frac{\alpha^{i}(-1)^{i}}{i!}\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{i\beta}$$

Equation (22) can be rewritten as

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi) = \alpha\beta\lambda\sum_{i=0}^{\infty}\frac{\alpha^{i}(-1)^{i}}{i!}\frac{g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{[G(x;\xi)]^{\lambda+1}}\left[\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right]^{-1}$$

$$\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta(1+i)-1}$$

$$(23)$$

But

$$\left(-\log\left(1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)\right)^{\beta(i+1)-1} = \left\{\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{\beta(i+1)-1} + \left(\beta(i+1)-1\right)\right.$$

$$\times\sum_{j=0}^{\infty}p_{j}\left((\beta(i+1)-1)+j\right)\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{j+\beta(i+1)}\right\}$$

Expanding $\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{\beta(i+1)-1}$ and $\left(\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right)^{j+\beta(i+1)}$ in power series

Equation (23) can be expressed as

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi)=\alpha\beta\lambda\sum_{i=0}^{\infty}\frac{\alpha^{i}(-1)^{i}}{i!}\frac{g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{\left[G(x;\xi)\right]^{\lambda+1}}\left[\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right]^{-1}$$

$$\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}(\beta(i+1)-1)^{k}}{k!}\left(\frac{1-G(x)}{G(x)}\right)^{\lambda k}+\left(\beta(i+1)-1\right)\sum_{j=0}^{\infty}p_{j}((i+1)\beta+j)\sum_{k=0}^{\infty}\frac{(-1)^{k}(j+\beta(i+1))^{k}}{k!}\left(\frac{1-G(x)}{G(x)}\right)^{\lambda k}\right\}$$

But

$$\left[\exp\left[\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]-1\right]^{-1}=\frac{\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]}{\left[1-\exp\left[-\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda}\right]\right]}$$

(24)

Using Binomial expansion $(x+a)^{-n}=\sum_{i=0}^{\infty}\binom{-n}{i}x^{i}a^{-n-i}=\sum_{i=0}^{\infty}(-1)^{i}\binom{n+k-1}{i}x^{i}a^{-n-i}$ to the denominator and power series expansion to the numerator of equation (24), we have:

$$f_{W-OFG}(x;\alpha,\beta,\lambda,\xi)=\alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{i}(-1)^{i+m+n}(1+i)^{n}}{i!n!}\frac{g(x;\xi)[1-G(x;\xi)]^{\lambda-1}}{\left[G(x;\xi)\right]^{\lambda+1}}\left(\frac{1-G(x;\xi)}{G(x;\xi)}\right)^{\lambda n}$$

$$\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}(\beta(i+1)-1)^{k}}{k!}\left(\frac{1-G(x)}{G(x)}\right)^{\lambda k}+\left(\beta(i+1)-1\right)\sum_{j=0}^{\infty}p_{j}((i+1)\beta+j)\sum_{l=0}^{\infty}\frac{(-1)^{k}(j+\beta(i+1))^{k}}{k!}\left(\frac{1-G(x)}{G(x)}\right)^{\lambda k}\right\}$$

This can further be simplify as

$$f_{W-OFG}(x,\alpha,\beta,\lambda,\xi)=\alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\alpha^{j}(-1)^{i+m+n}(1+i)^{n}}{i!n!}\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}(\beta(i+1)-1)^{k}}{k!}+\left(\beta(i+1)-1\right)\sum_{j=0}^{\infty}p_{j}((i+1)\beta+j)\right.$$

$$\times\sum_{l=0}^{\infty}\frac{(-1)^{k}(j+\beta(i+1))^{k}}{k!}\left\}\frac{g(x;\xi)[1-G(x;\xi)]^{\lambda(1+k+n)-1}}{\left[G(x;\xi)\right]^{\lambda(1+k+n)+1}}$$

(25)

But

$$[1-G(x;\xi)]^{\lambda(1+k+n)-1}=\sum_{q=0}^{\infty}(-1)^{q}\binom{\lambda(1+k+n)-1}{q}G(x;\xi)^{q}$$

Then equation (25) becomes

$$f_{W-OFG}(x,\alpha,\beta,\lambda,\xi)=\alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\frac{\alpha^{j}(-1)^{i+m+n+q}(1+i)^{n}}{i!n!}\binom{\lambda(1+k+n)-1}{q}\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}(\beta(i+1)-1)^{k}}{k!}+(\beta(i+1)-1)\sum_{j=0}^{\infty}p_{j}((i+1)\beta+j)\right.$$

$$\left.\times\sum_{l=0}^{\infty}\frac{(-1)^{k}(j+\beta(i+1))^{k}}{k!}\right\}g(x;\xi)\left[G(x;\xi)\right]^{q-\lambda(1+k+n)+1}$$

(26)

Equation (26) can be rewritten as

$$f(x;\alpha,\beta,\lambda,\xi)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\Phi_{k,n,q}h_{(q-\lambda(1+k+n)+2)}(x)$$

(27)

Where

$$h_{(q-\lambda(1+k+n)+2)}(x)=(q-\lambda(1+k+n)+2)g(x;\xi)G(x;\xi)^{(q-\lambda(1+k+n)+1)}$$

And

$$\Phi_{k,n,q}=\alpha\beta\lambda\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{i}(-1)^{i+m+n+q}(1+m)^{n}}{i!n!(q-\lambda(1+k+n)+2)}\binom{\lambda(1+k+n)-1}{q}\left\{\sum_{k=0}^{\infty}\frac{(-1)^{k}(\beta(i+1)-1)^{k}}{k!}+(\beta(i+1)-1)\sum_{j=0}^{\infty}p_{j}((i+1)\beta+j)\right.$$

$$\left.\times\sum_{k=0}^{\infty}\frac{(-1)^{k}(j+\beta(i+1))^{k}}{k!}\right\}$$

(28)

And the corresponding cdf is given by

$$F(x;\alpha,\beta,\lambda,\xi)=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\Omega_{l,m}G(x)^{m-l\lambda}$$

(29)

Where

$$\Omega_{l,m}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+2+l+m}\alpha^{(i+1)}}{(i+1)!l!}\binom{l\lambda}{m}\left\{(\beta(i+1))^{l}+(i+1)\beta\sum_{j=0}^{\infty}p_{k}((i+1)\beta+k)(k+(i+1)\beta+1)^{l}\right\}$$

(30)

And $G(x;\xi)$ is a baseline cdf, which depends on a parameter vector ξ.

3.2 Mathematical properties of the proposed family

In this section, we provide some structural properties of new Generalized Weibull- Odd Frèchet (NGWoFrG) family of distributions.

3.2.1 Moments

The rth moments of the proposed family can be obtained from equation (27) as

$$E(X^r) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\Phi_{k,n,q}E(Z_{k,n,q}^r) \tag{31}$$

Where $Z_{k,n,q}$ denotes the exponential-G distribution with power parameter $\left(q-\lambda(1+k+n)+1\right)$. Since the inner quantities in pdf (27) are absolutely integrable, the incomplete moments of X can be written as

$$I_X(y) = \int_{-\infty}^{y} x^r f(x;\alpha,\beta,\lambda)dx = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\Phi_{k,n,q}I_{k,n,q}(y)$$

Where

$$I_{k,n,q}(y) = \int_{-\infty}^{y} x^r h_{(q-\lambda(1+k+n)+2)}(x;\xi)dx$$

3.2.2 Moment generating function

The moments generating function of the proposed family is defined as;

$$M_x(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\Phi_{k,n,q}E(e^{tz_{k,n,q}}) \tag{32}$$

Where

$$E(e^{tz_{k,n,q}}) = \int_{0}^{\infty} e^{tz_{k,n,q}} h_{(q-\lambda(1+k+n)+2)}(z;\xi)dz$$

3.2.3 Entropies

The Renyi entropy of a random variable X represents a measure of variation of the uncertainty. The Renyi entropy is defined by:

$$I_\theta(x) = \frac{1}{1-\theta}\log\int_{-\infty}^{\infty} f(x)^\theta dx, \theta > 0 \text{ and } \theta \neq 1$$

Using the pdf of our proposed class of distributions (27), we can write

$$f(x)^\theta = \left(\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\Phi_{k,n,q}h_{(q-\lambda(1+k+n)+2)}(x)\right)^\theta$$

Then, the Renyi entropy of a random variable X having the New Generalized Weibull- odd Frèchet Family of Distributions is given by

$$I_\theta(x) = \frac{1}{1-\theta} \log \left\{ \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Phi_{k,n,q} \right)^\theta \int_{-\infty}^{\infty} \left(h_{(q-\lambda(1+k+n)+2)}(x;\xi) \right)^\theta dx \right\}; \theta > 0 \text{ and } \theta \neq 1 \quad (33)$$

The r-entropy, say $H_r(x)$, is given by

$$H_r(x) = \frac{1}{r-1} \log \left\{ 1 - \int_{-\infty}^{\infty} f(x)^r dx \right\}; r > 0 \text{ and } r \neq 1$$

By using the pdf of our proposed class of distributions, we can write

$$f(x)^r = \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Phi_{k,n,q} h_{(q-\lambda(1+k+n)+2)}(x) \right)^r$$

Then the r- entropy of a random variable X having the New Generalized Weibull- odd Frèchet Family of Distributions is given by

$$\left\{ \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Phi_{k,n,q} \right)^r \int_{-\infty}^{\infty} \left(h_{(q-\lambda(1+k+n)+2)}(x;\xi) \right)^r dx \right\}$$

$$H_q(x) = \frac{1}{q-1} \log \left\{ 1 - \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Phi_{k,n,q} \right)^r \int_{-\infty}^{\infty} \left(h_{(q-\lambda(1+k+n)+2)}(x;\xi) \right)^r dx \right\}; r > 0 \text{ and } q \neq 1 \quad (34)$$

3.2.4 Order statistics

Let $X_1, X_2,..., X_n$ be a random sample from our proposed class of distributions and $X_{1:n} \leq X_{2:n} \leq ... X_{n:n}$ denote the corresponding order statistics. $f_{i:n}(x)$ denote, the pdf of the ith order statistics $X_{i:n}(x)$, then the order statistics for our proposed class of distribution is obtained as:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x;\alpha,\beta,\lambda,\xi) F(x;\alpha,\beta,\lambda,\xi)^{i-1} [1 - F(x;\alpha,\beta,\lambda,\xi)]^{n-i}$$

But $[1 - F(x;\alpha,\beta,\lambda,\xi)]^{n-i} = \sum_{j=0}^{i-1} (-1)^j \binom{n-i}{j} F(x;\alpha,\beta,\lambda,\xi)^j$

Then we have

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{i-1} (-1)^j \binom{n-i}{j} \left(\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \Omega_{k,m} G(x;\xi)^{m-\lambda k} \right)^{i+j-1} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \Phi_{k,n,q} h_{(q-\lambda(1+k+n)+2)}(x)$$

(35)

Where $h_{(q-\lambda(1+k+n)+2)}(x)$ is the Exponential-G family with power parameter $(q-\lambda(1+k+n)+1)$, $G(x;\xi)$ is the baseline cdf, $\Omega_{l,m}$ and $\Phi_{k,n,q}$ are respectively given in equations (30) and (28).

4 Parameter Estimation

4.1 Maximum likelihood method

The maximum likelihood method is the most common method employed in most of the research for parameter estimate, so we consider the estimation of the unknown parameters of our proposed class of distributions from complete samples only by maximum likelihood estimates. Let $x_1, x_2, ..., x_n$ be the observed values from the New Generalized Weibull- odd Frèchet Family of Distributions with parameters α, β, λ and ξ. Let $\eta = (\alpha, \beta, \lambda, \xi)^T$ be the $P \times 1$ parameter vector. Then the log-likelihood function of η is given by

$$l(\eta) = n\log\alpha + n\log\beta + n\log\lambda + \sum_{i=1}^{n} \log g(x_i;\xi) - (\lambda+1)\sum_{i=1}^{n} \log G(x_i;\xi) + (\lambda-1)\sum_{i=1}^{n} \log \bar{G}(x_i;\xi) - \sum_{i=1}^{n} \left(\left[(H_i(x_i;\xi))^{\lambda} \right] - 1 \right)$$

$$+ (\beta-1)\sum_{i=1}^{n} \log\left(-\log\left(1 - \exp\left[-(H_i(x;\xi))^{\lambda} \right] \right) \right) - \alpha\sum_{i=0}^{n} \left(-\log\left(1 - \exp\left[-(H_i(x;\xi))^{\lambda} \right] \right) \right)^{\beta}$$

Where

$$H_i(x;\xi) = \frac{\bar{G}(x;\xi)}{G(x;\xi)}.$$ The components of the score function:

$$U(\eta) = \frac{\partial l}{\partial \eta} = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \xi} \right)^T = \left(U_{\alpha}, U_{\beta}, U_{\lambda}, U_{\xi_r} \right)^T,$$ are given by

$$U_{\alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n} \left(-\log\left(1 - \exp\left[-(H_i(x;\xi))^{\lambda} \right] \right) \right)^{\beta}$$

$$U_{\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log\left(-\log\left(1 - \exp\left[-(H_i(x;\xi))^{\lambda} \right] \right) \right) - \alpha\beta\sum_{i=0}^{n} \left(-\log\left(1 - \exp\left[-(H_i(x;\xi))^{\lambda} \right] \right) \right)^{\beta-1}$$

$$U_{\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \log G(x_i;\xi) - \sum_{i=1}^{n} \log \bar{G}(x_i;\xi) - \sum_{i=1}^{n} \left(s_i^{(\lambda)} \right) + (\beta-1)\sum_{i=1}^{n} \log\left(-\log\left(1 - z_i^{(\lambda)} \right) \right) - \alpha\sum_{i=0}^{n} \left(-\log\left(1 - z_i^{(\lambda)} \right) \right)^{\beta}$$

Where

$$s_i = \left(\left[(H_i(x_i;\xi))^{\lambda} \right] - 1 \right)$$

And

$$z_i = \exp\left[-\left(H_i(x;\xi)\right)^{\lambda}\right]$$

Equating $U_{\alpha}, U_{\beta}, U_{\lambda}, U_{\xi_k}$ (for k=1…, p) to zero and solving the equations simultaneously yields the MLE $\hat{\eta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\xi}_k)^T$ of $\eta = (\alpha, \beta, \lambda, \xi_k)^T$.

Almost all the analysis (results) obtained in this work cannot be achieving without this section especially in obtaining the goodness of fit measures. All the derivation in this section cannot be solve numerically, so we adopt the statistical software called R-studio in R-package to obtain the estimate of the model parameters. These estimated parameters available in Tables 2 and 3 maximize the likelihood that the process described by this model will produced the data that were actually observed.

5 Monte Carlo Simulation and Application

5.1 Simulation study

Construction of model depend heavily on assumptions which associated with uncertainty, Monte Carlo simulation help to explain the impact of risk and uncertainty in prediction and forecasting models. In this work we adopt the Monte Carlo simulation to ascertain our proposed distribution about the risk in modeling lifetime data.

To assess the performance of NGWoFr family distribution, the simulation study was conducted using the Monte Carlo Simulation method to compute the mean, bias and variance of the estimated parameters from the maximum likelihood estimates. The Simulated data is generated using the quantile function defined in equation (21) for different sample size n= 50, 75, and 100 and replicated 100 times. For each sample size $\alpha = 3.8$, $\beta = 1.9$, $\lambda = 1$, $\theta = 0.1$ and $\gamma = 0.262$.

Table 1. Presents the results of bias and variance from the simulated distribution

Sample sizes (n)	Parameters	Estimates	Bias	Variance
50	$\hat{\alpha}$	0.09031714	-3.70968300	0.00527840
	$\hat{\beta}$	5.02550200	3.01255020	0.00036323
	$\hat{\lambda}$	0.34417260	-0.65582740	0.00140305
	$\hat{\theta}$	0.26031930	0.16031930	0.00164769
	$\hat{\gamma}$	0.07633079	-0.18566920	0.00272111
75	$\hat{\alpha}$	0.05659038	-3.74341000	0.001628607
	$\hat{\beta}$	5.04138200	3.11413820	0.00024201
	$\hat{\lambda}$	0.33088120	-0.66911880	0.00039686
	$\hat{\theta}$	0.25413510	0.15413510	0.00067897
	$\hat{\gamma}$	0.07799147	-0.18400850	0.00159897
100	$\hat{\alpha}$	0.02719936	-3.77280100	0.00072597
	$\hat{\beta}$	5.05893900	3.15893900	0.00018228
	$\hat{\lambda}$	0.31743060	-0.68256940	0.00020797
	$\hat{\theta}$	0.24972440	0.14972440	0.00046607
	$\hat{\gamma}$	0.07972022	-0.18227980	0.00109517

Table 1 represents the results obtained from the Monte Carlo Simulation study. These results show that the bias and variance decreases toward zero with an increase in sample size, this indicates that the proposed distribution is suitable and less risk in modeling lifetime dataset.

5.2 Applications

In this section, we demonstrate the significance of New Generalized Weibull- odd Frèchet–Frechet Distribution using two real datasets. The maximum likelihood estimates, as well as goodness-of-fit measures, are computed and compared with these competing models:

(a) Odd Log-Logistic- Frechet distribution (OLL-FrD) defined by Haitham et al. [16] with cdf

$$F(x;\alpha,\theta,\gamma) = \frac{exp\left[-\alpha\left(\frac{\theta}{x}\right)^{\gamma}\right]}{exp\left[-\alpha\left(\frac{\theta}{x}\right)^{\gamma}\right] + \left(1-exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\right)^{\alpha}}, \quad where \quad x>0, \ \alpha>0, \ \gamma>0 \ and \ \theta>0,$$

And the corresponding pdf

$$f(x;\alpha,\theta,\gamma) = \alpha\theta^{\gamma}\gamma x^{-(\gamma+1)}exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\left(exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\left(1-exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\right)\right)^{\alpha-1}$$

$$\times\left(exp\left[-\alpha\left(\frac{\theta}{x}\right)^{\gamma}\right] + \left(1-exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\right)^{\alpha}\right)^{-2}$$

where $x \geq 0, \ \alpha > 0, \ \gamma > 0 \ and \ \theta > 0,$ With scale parameters $\alpha > 0$

While $\theta > 0, \ \gamma > 0$ are the shape parameters.

(b) Alpha Logarithm Transformed Frechet distribution (ALT-FrD) defined by Sanku et al. [17] with cdf

$$F(x;\lambda,\theta,\gamma) = 1 - \frac{\log\left(\lambda-(\lambda-1)exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\right)}{\log(\lambda)}, \qquad where \ x>0, \ \lambda>0, \ \gamma>0 \ and \ \theta>0,$$

And the corresponding pdf

$$(x; \lambda, \theta, \gamma) = \frac{(\lambda - 1)\theta^{\gamma} \gamma x^{-(\gamma+1)} exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]}{\log(\lambda)\left(\lambda - (\lambda - 1)exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]\right)}$$

With scale parameters $\theta > 0$ while $\lambda > 0$ *and* $\gamma > 0$ are the shape parameters.

(c) Frechet distribution (FrD) defined by Maurice Frechet [12] with cdf

$$F(x; \theta, \gamma) = exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right], \qquad where\ x > 0,\ \gamma > 0\ \ and\ \ \theta > 0,$$

And the corresponding pdf

$$f(x; \theta, \gamma) = \theta^{\gamma} \gamma x^{-(\gamma+1)} exp\left[-\left(\frac{\theta}{x}\right)^{\gamma}\right]$$

With scale parameters $\theta > 0$ and $\gamma > 0$ as the shape parameter

Data 1: Data set relates to tests on endurance of deep groove ball bearings.

The first data set represents the number of million revolutions before failure for each of the 23 ball bearings in the life test by [18] and they are:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40

For the endurance of deep groove ball bearings data, we fit and compared the performances of the NGWF-Fr with OLL-Fr, ALT-Fr, and Fr distributions by comparing the Akaike Information Criterion (AIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) statistic using the Adequacy model package in R.

Table 2. Parameters estimates and goodness of fit measures for first data set

Models	Parameter estimates					Goodness of fit		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$	AIC	BIC	HQIC
NGWFr-FrD	0.244045	1.858217	1.060520	1.804910	0.526641	278.280	283.957	279.708
OLL-FrD	1.797290	1.763615	0.191687	-	-	305.6163	309.0228	306.4730
ALT-FrD	1.924226	0.309823	1.514723	-	-	292.5620	295.9685	293.4187
FrD	-	-	-	1.802343	0.339246	314.2031	316.4741	314.7742

Table 2 provides the parameters estimate and goodness of fit measures for the New Generalized Weibull-odd Frèchet- Frèchet distribution (NGWoFr-FrD) and other competitor distributions.
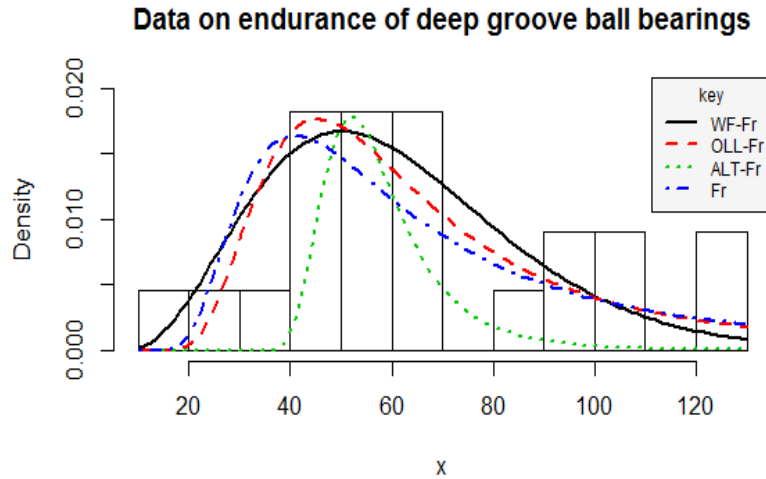
Fig. 2. Deep groove ball bearings (histogram) and the fitted distributions

From Table 2 and Fig. 2 it is clear that the new Generalized Weibull- odd Frèchet- Frèchet distribution (NGWoF-FrD) performs better than its competitors.

Data 2: Relates to the intervals of times at which vehicles pass a point on a road.

The second data presents the length of intervals between the times at which vehicles pass a point on a road. The data source is Jørgensen [19]. The observations are as follows: 2.50, 2.60, 2.60, 2.70, 2.80, 2.80, 2.90, 3.00, 3.00, 3.10, 3.20, 3.40, 3.70, 3.90, 3.90, 3.90, 4.60, 4.70, 5.00, 5.60, 5.70, 6.00, 6.00, 6.10, 6.60, 6.90, 6.90, 7.30, 7.60, 7.90, 8.00, 8.30, 8.80, 8.80, 9.30, 9.40, 9.50, 10.1, 11.0, 11.3, 11.9, 11.9, 12.3, 12.9, 12.9, 13.0, 13.8, 14.5, 14.9, 15.3, 15.4, 15.9, 16.2, 17.6, 20.1, 20.3, 20.6, 21.4, 22.8, 23.7, 23.7, 24.7, 29.7, 30.6, 31.0, 34.1, 34.7, 36.8, 40.1, 40.2, 41.3, 42.0, 44.8, 49.8, 51.7, 55.7, 56.5, 58.1, 70.5, 72.6, 87.1, 88.6, 91.7, 119.8.

For the second data, we evaluate the performances of new Generalized Weibull- odd Frèchet- Frèchet distribution (NGWoF-FrD) with the same competitive distributions used for the first dataset.

Table 3. Parameters estimates and goodness of fit measures for first data set

Models	Parameter estimates					Goodness of fit		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\gamma}$	AIC	BIC	HQIC
NGWF-FrD	0.869319	1.972623	0.696449	1.399457	0.675642	701.8953	714.0494	706.7812
OLL-FrD	1.962117	1.927482	0.421730	-	-	739.2425	746.5349	742.174
ALT-FrD	0.956286	0.490810	0.908457	-	-	828.896	836.1885	831.8275
FrD	-	-	-	1.944688	0.679438	770.2945	775.1561	772.2488

Table 3 provides the parameters estimate and goodness of fit measures for the new Generalized Weibull-odd Frèchet- Frèchet distribution (NGWoF-FrD) and other competitor distributions.
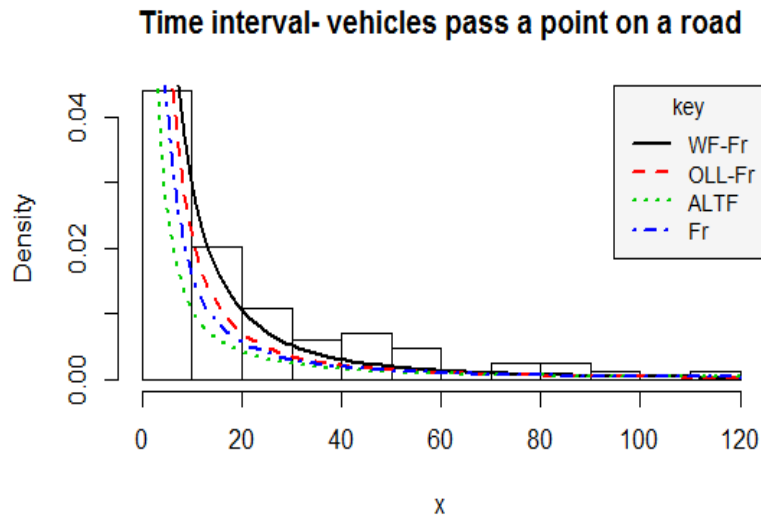
Fig. 3. Time interval (histogram) and the fitted distributions

From Table 3 and Fig. 3 it is clear that the new Generalized Weibull- odd Frèchet- Frèchet distribution (NGWoF-FrD) performs better than its competitors.

6 Conclusion

Generalizing a standard distribution provides more flexibility in modeling real data. We introduced a family of distributions called new Generalized Weibull- Odd Frèchet (NGW-oFr) family of distributions in order to provide more flexible distribution. We provide some structural properties of the new family including shapes, moments and incomplete moments, moment generating functions, order statistic and Renyi entropies. The model parameters are estimated by maximum likelihood method and Monte Carlo simulation is used to test the robustness of the parameters. The sub-model of the family was fitted to two real datasets to illustrate its flexibility. The result indicates that the sub-model of the family provides better fit than the other competitors for modeling the two data sets. This new family has unlimited extension, it can generalize any existing continuous distribution by adding three parameters, each of these generalized distributions can further be study by designing the confidence interval for estimated parameters or estimating the parameters using Bayesian approach.

Competing Interests

Authors have declared that no competing interests exist.

References

[1]   Merovci F. Transmuted Rayleigh distribution. Austrian Journal of Statistics. 2013b;42(1):21-31.

[2]   Gupta RD, Kundu D. Generalized exponential distributions. Australian and New Zealand Journal of Statistics. 1999;41(2):173–188.

[3]   Eugene N, Lee C, Famoye F. Beta-normal distribution and it applications. Communication in Statistics: Theory and Methods. 2002;31:497-512.

[4]   Cordeiro GM, de Castro M. A new family of generalized distributions. Journal of Statistical Computation and Simulation. 2011;81(7):883-898.

[5]   Cordeiro GM, Ortega EM, da Cunha DC. The exponentiated generalized class of distributions. Journal of Data Science. 2013;11:1-27.

[6]   Alzaatreh A, Famoye F, Lee C. The Weibull-Pareto distribution and its applications. Communications in Statistics-Theory and Methods. 2013;42:1673–1691.

[7]   Cordeiro GM, Ortega EMM, Popovic BV, Pescim RR. The Lomax generator of distributions: Properties, magnification process and regression model. Applied Mathematics and Computation. 2014;247:465-486.

[8]   Alizadeh M, Cordeiro GM, Debrito E, Demetrio CGB. The Betamarshall-Olkin family of distributions. Journal of Statistical Distributions and Applications. 2015a;2(4):1-18.

[9]   Alizadeh M, Tahir MH, Cordeiro GM, Mansoor M, Zubair M, Hamedani GG. The Kumaraswamy Mashal-Olkin family of distributions. Journal of the Egyptian Mathematical Society. 2015;23(3):546-557.

[10]   Afify AZ, Cordeiro GM, Yousof HM, Alzaatreh A, Nofal ZM. The Kumaraswamy Transmuted-G family of distributions: Properties and applications. J. Data Sci. 2016;14:245-270.

[11]   Weibull W. Wide applicability. Journal of Applied Mechanics. 1951;18:293-297.

[12]   Frechet M, Surla L. de Probabilited elecart maximum. Annales de la Societe Polonaise de Mathematique, Cracovie. 1927;6:93-116.

[13]   Harlow DG. Applications of the Frechet distribution function. International Journal of Material and Product Technology. 2002;5(17):482-495.

[14]   Cordeiro GM, Ortega MM, Ramires TG. A new generalized Weibull family of distributions: Mathematical properties and applications. Journal of Statistical Distributions and Applications. 2015;13(2):1-25.

[15]   UlHaq AM, Elgarhy M. The Odd Frèchet-G family of probability distributions. Journal of Statistics Applications & Probability. 2018;7(1):185-201.

[16]   Haitham MY, Emrah A, Hamedani GG. A new extension of Frechet distribution with regression models, residual analysis and characterizations. Journal of Data Science. 2018;16(4):743-770.

[17]   Sanku D, Mazen N, Devendra K, Fahad A. Alpha logarithm transformed Frechet distribution: Properties and estimation. Austrian Journal of Statiatics. 2019;48:70-93.

[18]   Linhart H, Zucchini W. Model selection. John Wiley, New York; 1986.

[19]   Jørgensen B. Statistical properties of the generalized inverse Gaussian distribution. New York: Springer-Verlag. MR0648107; 1982.