

Numerical Solutions for Solving the Modeling Differential Equations

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Authors' contributions

This work was carried out in collaboration between all authors. Authors YAA and AMSM designed the study, performed the numerical analysis, wrote the protocol and wrote the first draft of the manuscript.

Authors AMSM and HARN managed the analyses of the study. Author AMSM managed the literature searches. All authors read and approved the final manuscript.

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Abstract

In this paper, we will solve the Logistic and Riccati differential equations using *VIM*, shifted Chebyshev-spectral fourth kind methods and Hermite collocation method. Where we can from the numerical results we obtained to conclude that the solution using these three approaches converge to the exact solution is excellent. We note that we can apply the proposed methods to solve other problems in engineering and physics.

Keywords: Logistic differential equation; Riccati differential equation; Variational iteration method; Shifted Chebyshev-spectral method fourth kind; Hermite collocation method.

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1 Introduction

There are a number of analytical and approximation methods to solve the nonlinear differential equation of between it **VIM** and the method is present by He [1]. There are useful differences and correction of functions to solve nonlinear equations [2-8]. In differential equations does not require the presence of small parameters, and not to be heterogeneous with respect to the dependent variable and its derivatives. This technique provides parity in functions that meet the exact solution of the problem. The problem is solved without the need for discretization of variables. Therefore, in some problems, the round of errors is not affected by one error that you do not experience with large computer memory. The proposed scheme provides a solution to the problem in a closed form, the difference method such as Limited [9] provides approximation in grid points only. This procedure is a powerful tool to solve various kinds of problems, such as, as used to solve the delay differential equations in [10], with these advantages of VIM corresponded to some negatives for example, the main objective of this paper is to introduce a new amendment to this method to overcome defects and increase the rate of convergence of this method. Methods of solution based on orthogonal polynomials known as spectral methods. For example, the spectral method of non-linear high-grade nonlinear differential equations [11,12], the spectral method to the fractional diffusion equation [13]. The spectral method has proven itself to be the most suitable for computer execution.

In this paper, we use the spectral method with the recruitment of the Chebyshev-collocation points, to obtain the solution of the differential equation of numerical logistics is very accurate numerical. The logistic model of the differential equation is continuous at the time it is described as the normal differential equation. Hermite polynomials are widely used in numerical computation. One of the advantages of using Hermite polynomials as a tool for expansion functions is the good representation of smooth functions by finite Hermite expansion provided that the function $u(t)$ is infinitely differentiable. The logistic model was proposed by the Belgian mathematician Pierre Verhulst in 1838 [14]. There are many variations in population modeling [15-17]. A classic example of anarchic behavior in a dynamic system [14]. The model shows population growth and population density [18]. the solution puts the rate of population growth constant and does not include curbing the spread of disease and food supply. The curve of the solution increases exponentially and is the maximum absorptive capacity [18], N is the population, r is the rate of population growth and K is the carrying capacity.

The Riccati differential equation (*RDE*) is named after the Italian Jacopo Francesco Riccati (1676-1754). Reed book contains [19-25] On the basic theory of the Riccati equation, With applications for random operations, Optimal control, Propagation problems and applications of important engineering sciences today are considered a classic, strong stability, optimal control, network synthesis, applications include the latest in areas such as financial mathematics [26]. This equation can be solved using the classical numerical method such as, the Euler method, the Rung Kutta method, and the Bahnasawi and others in [25-28].

The main goal in this article is concerned with the application of VIM and Chebyshev-spectral and Hermite methods to obtain the numerical solution of the Logistic Riccati, differential equation of the form The Logistic differential equation [24]

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t)), \quad t > 0, \quad \rho > 0 \quad (1)$$

We also assume an initial condition

$$u(0) = u_0, \quad u_0 > 0. \quad (2)$$

The exact solution to this problem is given by $u(t) = \frac{u_0}{(1-u_0)e^{-\rho t} + u_0}$.

The Riccati differential equation

$$\frac{du(t)}{dt} + u^2(t) - 1 = 0, \quad t > 0, \quad (3)$$

We also assume an initial condition

$$u(0) = u^0. \quad (4)$$

The exact solution to this problem at $u^0 = 0$ is $u(t) = \frac{e^{2t}-1}{e^{2t}+1}$.

1.1 Applications of logistic differential equation

Is a model of population growth. Allow $u(t)$ population size and t time where ρ constant growth rate. The differential equation of logistics is used in medicine to model tumors. Consider this application as an extension mentioned above in ecology. $u(t)$ Size of tumors in time t .

There are applications in various fields of the logistics model among them;

- i. **Neural Networks:** Logistic Models are often used in neural networks to introduce nonlinearity in the model and or to clamp signals within a specific range. A popular neural net element computes a linear combination of its input signals, and applies bounded logistic function to the result; this model can be seen as a smoothed variant of the classical threshold neuron.
- ii. **Statistics:** Logistic functions are used in several roles in statistics; firstly, they are the cumulative distribution function of the logistic family of distribution. Secondly, they are used in logistic regression to model how the probability of an event may be affected by one or more explanatory variables.
- iii. **Chemistry:** the concentration of reactants and products in autocatalytic reactions follows the logistic function.
- iv. **Physics:** it is applied in Fermi distribution in the sense that the logistic function determines the statistical distribution of fermions over the energy states of a system in thermal equilibrium. In particular, it is the distribution of the probabilities that each possible energy level is occupied by fermions, according to Fermi Dirac statistics.
- v. **Linguistics:** in linguistic, the logistic function can be used to model language change, an innovation that was at first marginal but has now become more universally adopted.
- vi. **Economics:** the logistic function can be used to illustrate the progress of the diffusion of an innovation, infrastructures and energy source substitutions and the role of work in the economy as well as with the long economic cycle.

The paper is organized as follows: Section 2, we implement **VIM** -technique for solving non-linear Logistic differential equation. In Section 3, we implement **VIM**-technique for solving non-linear Riccati differential equation. In Section 4, solution procedure using the shifted chebyshev-spectral method fourth kind Logistic, Riccati differential equation. In section 5, we study some properties of the Hermite polynomials. In section 6, the Hermite method to solve numerically the non-linear Logistic, Riccati differential equation. In section 7, the paper ends with a brief conclusion.

The existence and uniqueness for the Logistic differential equation in [24].

The Convergence analysis for the Logistic equation of **VIM** is satisfy in [24]. The maximum absolute error for the Logistic equation of the approximate solution in [24].

2 VIM-technique to Non-linear Logistic Differential Equation

In this section, we implement *VIM*-technique for solving non-linear Logistic differential equation(1).

Step 1.Solve Eq.(1) by using VIM; We rewrite Eq.(1) in the following operator form

$$Lu = \rho(u - u^2), \tag{5}$$

Where $L = \frac{d}{dt}$ is linear bounded operator, i.e., it is possible to find number $K > 0$ such that $\|L_u\| \leq k \|u\|$. The *VIM* gives the possibility to write the solution of Eq.(5) with the aid of the correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) [\dot{u}_n + \rho(\tilde{u}_n^2 - \tilde{u}_n)] d\tau \tag{6}$$

Making the above correction functional stationary, and noticing that $\delta\tilde{u}_n = 0$ we obtain

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda [\dot{u}_n + \rho(\tilde{u}_n^2 - \tilde{u}_n)] d\tau \\ &= \delta u_n(t) + \int_0^t \lambda \delta \dot{u}_n d\tau = \delta u_n(t) + [\lambda(\tau) \delta u_n]_{\tau=t} - \int_0^t \dot{\lambda}(\tau) [\delta u_n] d\tau = 0 \end{aligned}$$

Where $\delta\tilde{u}_n$ is considered as a restricted variation i.e., $\delta\tilde{u}_n = 0$; yields the following stationary conditions

$$\dot{\lambda}(\tau) = 0, \quad 1 + \lambda(\tau) |_{\tau=t} = 0 \tag{7}$$

Eq.(7) is called Lagrange-Euler equation with its boundary condition. The Lagrange multiplier can be identified by solving this equation as $\lambda(\tau) = -1$. Now, the following variational iteration formula can be obtained

$$u_{n+1} = u_n(t) - \int_0^t [\dot{u}_n + \rho(u_n^2 - u_n)] d\tau, \quad n \geq 0 \tag{8}$$

We start with an initial approximation, and by using the iteration formula (8), we can obtain directly the other components of the solution. The successive approximations $u_n, n \geq 1$ of the solution u will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 . Consequently, the exact solution may be obtained by using

$$u = \lim_{n \rightarrow \infty} u_n. \tag{9}$$

Using the recurrence formula (8), we can obtain the components of the approximate solution of(1) [24].

$$\begin{aligned} u_0(t) &= 0.85, \\ u_1(t) &= 0.85 + 0.06375t, \\ u_2(t) &= 0.85 + 0.06375t - 0.0222t^2 - 0.00202t^3, \\ u_3(t) &= 0.85 + 0.06375t + 0.00974t^3 + 0.00205t^4 - 0.0001t^5 - 0.00004t^6 - 0.000002t^7, \end{aligned}$$

Therefore, the complete approximate solution can be readily obtained by the same iterative process. The behavior of the approximate solution using *VIM* is presented in Fig. 1.

From this figure, we can see that *VIM* is invalid when applied in a large domain or the error of this method is more large. So, in the next steps we present a modification in *VIM*; to improve the error.

Step 2. Truncate the sequence's solution obtained by **VIM**; we have applied the method by using four-iterations only, i.e., the approximate solution is

$$u(t) \cong u_4(t). \tag{10}$$

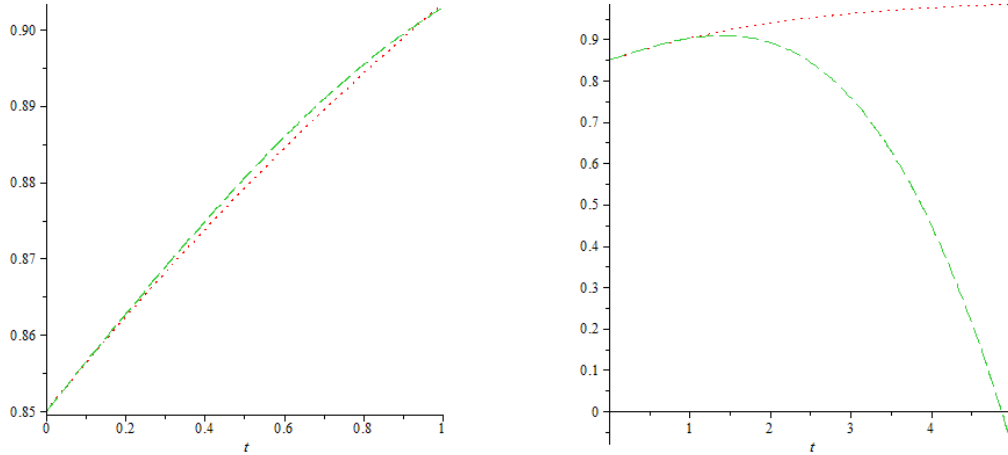


Fig. 1. Comparison of the approximate solution using **VIM** and the exact solution in $[0,1]$ and $[0, 5]$

3 VIM-technique to Non-linear Riccati Differential Equation

The procedure of the implementation is given by the following steps: In this section, we implement **VIM**-technique for solving non-linear Riccati differential equation (3)

Step 1. Solve Eq. (3) by using **VIM**; We rewrite Eq.(3) in the following operator form

$$Lu = 1 - u^2, \tag{11}$$

where $L = \frac{d}{dt}$ is linear bounded operator, i.e., it is possible to find number $K > 0$ such that $\| L_u \| \leq k \| u \|$

The **VIM** gives the possibility to write the solution of Eq.(11) with the aid of the correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau) [\dot{u}_n - 1 + \tilde{u}_n^2] d\tau \tag{12}$$

Making the above correction functional stationary, and noticing that $\delta \tilde{u}_n = 0$ we obtain

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda [\dot{u}_n - 1 + \tilde{u}_n^2] d\tau \\ \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda [\dot{u}_n - 1] d\tau \end{aligned}$$

Where $\delta \tilde{u}_n$ is considered as a restricted variation i.e., $\delta \tilde{u}_n = 0$, yields the following stationary conditions

$$\dot{\lambda}(\tau) = 0, \quad 1 + \lambda(\tau) |_{\tau=t} = 0 \tag{13}$$

Eq. (13) is called Lagrange-Euler equation with its boundary condition. The Lagrange multiplier can be identified by solving this equation as $\lambda(\tau) = -1$. Now, the following variational iteration formula can be obtained

$$u_{n+1}(t) = u_n(t) - \int_0^t [\dot{u}_n - 1 + u_n^2]d\tau, \quad n \geq 0 \tag{14}$$

We start with an initial approximation, and by using the iteration formula (14), we can obtain directly the other components of the solution. The successive approximations $u_n, n \geq 1$ of the solution u will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u^0 . Consequently, the exact solution maybe obtained by using

$$u = \lim_{n \rightarrow \infty} u_n. \tag{15}$$

Using the recurrence formula (14), we can obtain the components of the approximate solution of(3)

$$\begin{aligned} u_0(t) &= 0, \\ u_1(t) &= t, \\ u_2(t) &= t - 0.333333t^3, \\ u_3(t) &= t - 0.333333t^3 - 0.133333t^5 - 0.015873t^7, \\ u_4(t) &= t - 0.333333t^3 + 0.133333t^5 + 0.022222t^7 \\ &\quad - 0.006349t^9 - 0.002578t^{11}, \end{aligned}$$

4 Solution Procedure Using the Shifted Chebyshev-spectral Method Fourth Kind

In this section procedure using the Chebyshev-spectral method to solve numerically the Logistic differential equation (1).The well known Chebyshev polynomials [13,23] are defined on the interval [-1,1]

The Chebyshev polynomials $V_n(t)$ and $W_n(t)$ of the third and fourth kinds are polynomials in t defined, respectively, by [23]

$$V_n(t) = \frac{\cos\left(n + \frac{1}{2}\right)}{\cos\frac{\theta}{2}}$$

And

$$W_n(t) = \frac{\sin\left(n + \frac{1}{2}\right)}{\sin\frac{\theta}{2}}$$

where $t = \cos \theta$.

In fact the polynomials $V_n(t)$ and $W_n(t)$ are rescaling of two particular Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ for the two non symmetric special cases $\beta = -\alpha = \pm \frac{1}{2}$ They are given explicitly

By

$$V_n(t) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{\left(-\frac{1}{2}, \frac{1}{2}\right)}(t).$$

And

$$W_n(t) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{(\frac{1}{2}, -\frac{1}{2})}(t).$$

It is readily seen that

$$W_n(t) = (-1)^n W_n(-t).$$

In this section, we suggest $u(t)$ be expressed in chebyshev series

$$u(t) \approx \sum_{n=0}^m a_n W_n(t). \tag{16}$$

4.1 Chebyshev fourth kind for solving logistic differential equation

In this section, we introduce a discretization formula of (1) using the chebyshev collocation method to make this aim, we approximate $u(t)$ as

$$u_m(t) = \sum_{n=0}^m a_n W_n^*(t), \tag{17}$$

From Equations (1) and (16) we have

$$\sum_{n=0}^m a_n W_n^{*'}(t) = \rho \sum_{n=0}^m a_n W_n^*(t) \left(1 - \sum_{n=0}^m a_n W_n^*(t) \right). \tag{18}$$

We now collocate Eq. (18) at m points t_p as

$$\sum_{n=0}^m a_n W_n^{*'}(t_p) = \rho \sum_{n=0}^m a_n W_n^*(t_p) \left(1 - \sum_{n=0}^m a_n W_n^*(t_p) \right). \tag{19}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $W_m^*(t)$. Also, by substituting Eq. (17) in the given initial condition (2) we can find the following equation

$$\begin{aligned} u_m(t) &= \sum_{n=0}^m a_n W_n^*(t) \\ u_0 &= \sum_{n=0}^m a_n W_n^*(0) \\ u_0 &= \sum_{n=0}^m (-1)^n a_n \\ \sum_{n=0}^m (-1)^n a_n &= u_0. \end{aligned} \tag{20}$$

The system of Eqs. (19) - (20), is a non-linear system of $(m + 1)$ algebraic equations which can be solved, for the unknowns $a_n, n = 0, 1, \dots, m$ using a suitable method. Consequently $u(t)$ given in Eq. (1) can be calculated. In our computational we use the Newton's iteration method to solve the resulting non-linear system of algebraic equations.

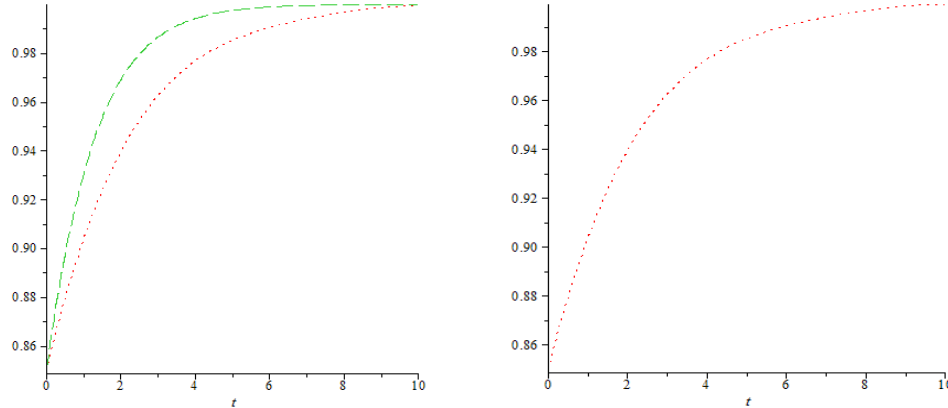


Fig. 2. Comparison of the exact solution and the approximate solution using VIM and Chebyshev-spectral method, where $u_0 = 0.85, \rho = 0.5$ with $m = 2$ (left) and $m = 4$ (right)

Now, to illustrate the applicability of the proposed method we implement the method with $m = 4$, and we approximate solution as

$$u(t) \approx \sum_{n=0}^4 a_n W_n(t). \tag{21}$$

Using Eq. (19) we have

$$\sum_{n=0}^4 a_n W_n^*(t_p) = \rho \sum_{n=0}^4 a_n W_n^*(t_p) \left(1 - \sum_{n=0}^4 a_n W_n^*(t_p) \right), \tag{22}$$

With $p = 0, 1, 2, 3$ where t_p are roots of the shifted Chebyshev polynomial $W_4^*(t)$ which is the approximate solution of the problem (1). The exact solution of the Logistic differential equation(1) is

$$u(t) = \frac{u_0}{(1 - u_0)e^{-\rho t} + u_0}. \tag{23}$$

The obtained numerical results by means of the Chebyshev-spectral method is shown in Figs. 2 and 3. Fig. 2 presents behavior of the approximate solution and the exact solution, where $u_0 = 0.85, \rho = 0.5$ at different values of m ($m = 2$ (left) and $m = 4$ (right)). Fig. 3 presents behavior of the approximate solution with the exact solution, at different values of u_0 and $u_0 = 0.85$ and $\rho = 0.2$

From these figures we can conclude that our approximate solution using Chebyshev-spectral method is in excellent agreement with the exact values.

4.2 Chebyshev fourth kind for solving Riccati differential equation

In this section, we introduce a discretization formula of (3) using the chebyshev collocation method to make this aim, we approximate $u(t)$ as From Equations (3) and (16) we have

$$\sum_{n=0}^m a_n W_n^{*'}(t) = 1 - \left(\sum_{n=0}^m a_n W_n^*(t) \right)^2 \tag{24}$$

We now collocate Eq. (24) at m points t_p as

$$\sum_{n=0}^m a_n W_n^{*'}(t_p) = 1 - \left(\sum_{n=0}^m a_n W_n^*(t_p) \right)^2 \tag{25}$$

For suitable collocation points we use roots of shifted Chebyshev polynomial $W_m^*(t)$. Also, by substituting Eq.(17) in the given initial condition (4) we can find the following equation

$$\begin{aligned} u^0 &= \sum_{n=0}^m a_n W_n^*(0) \\ u^0 &= \sum_{n=0}^m (-1)^n a_n \\ \sum_{n=0}^m (-1)^n a_n &= u^0. \end{aligned} \tag{26}$$

The system of Eqs. (25) - (26), is a non-linear system of $(m + 1)$ algebraic equations which can be solved, for the unknowns $a_n, n = 0, 1, \dots, m$, using a suitable method.

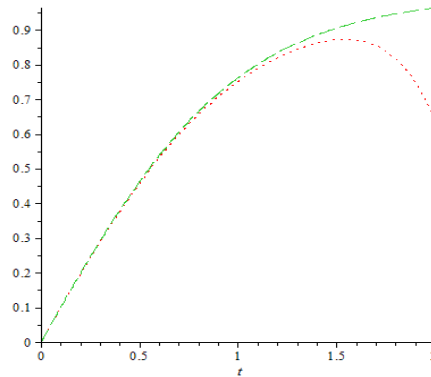


Fig. 3. Comparison of the exact solution and the approximate solution using Chebyshev-spectral method, where $u_0 = 0.85, \rho = 0.2$

Consequently $u(t)$ given in Eq.(3) can be calculated. In our computational we use the Newton's iteration method to solve the resulting non-linear system of algebraic equations.

Now, to illustrate the applicability of the proposed method we implement the method with $m = 4$, and we approximate solution as Eq. (21) Using Eq. (25) we have

$$\sum_{n=0}^4 a_n W_n^{*'}(t_p) = 1 - \left(\sum_{n=0}^4 a_n W_n^*(t_p) \right)^2 \tag{27}$$

With $p = 0,1,2,3$ where t_p are roots of the shifted Chebyshev polynomial $W_4^*(t)$.

5 Some Properties of the Hermite Polynomials

In this section, the main aim of the presented paper is concerned with the application of the Hermite collocation method to introduce the numerical simulation of the Logistic, Riccati differential equation. In mathematics, the Hermite polynomials are classical orthogonal polynomial sequence that arise in probability, such as the Edge worth series; in combinatorics, as an example of an Appell sequence, obeying the umbral calculus, in numerical analysis as Gaussian quadrature; in finite element methods as shape functions for beams, and in physics, where they give rise to the eigen states of the quantum harmonic oscillator. They are also used in systems theory in connection with nonlinear operations on Gaussian noise. They were defined by Laplace (1810) though in scarcely recognizable form, and studied in detail by Chebyshev (1859). Chebyshev's work was overlooked and they were named later after Charles Hermite who wrote on the polynomials in (1864) describing them as new. They were consequently not new although in later (1865) Hermite was the first to define the multidimensional polynomials [29-35].

Definition: The Hermite polynomials are given by [29]

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}$$

Some main properties of these polynomials are:

The Hermite polynomials evaluated at zero argument $H_n(0)$ and are called Hermite number as follows: [29]

$$H_n(0) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}} 2^{\frac{n}{2}} (n-1)!! & \text{if } n \text{ is even,} \end{cases} \quad (28)$$

Where $(n-1)!!$ is the double factorial. The polynomials $H_n(t)$ are orthogonal with respect to the weight function $\omega(t) = e^{-t^2}$ with the following condition:

$$\int_{-\infty}^{\infty} H_n(t) H_m(t) \omega(t) dt = \sqrt{\pi} 2^n n! \delta_{nm}.$$

In this article, the principal thought about the unearthy collocation strategy is to acknowledge that the dark result $u(t)$ could be approximated by a straight blend of a couple of reason limits, called the trial limits, for instance, orthogonal polynomials, as

$$u(t) \cong \sum_{i=0}^n c_i \varphi_i(t).$$

6 An Approximate Formula of the Fractional Derivative

The Hermite polynomials are defined on \mathbb{R} and can be determined with the aid of the following recurrence formula [29-34].

$$H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t), \quad H_0(t) = 1, \quad H_1(t) = 2t, \quad n = 1, 2, \dots$$

The analytic form of the Hermite polynomials of degree n is given by

$$H_n(t) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^{n-2k}}{(k)! (n-2k)!} t^{n-2k}. \tag{29}$$

In consequence, for the p-th derivatives of Hermite polynomials the following relation hold:

$$H_n^{(p)}(t) = 2^p \frac{n!}{(n-p)!} H_{n-p}(t) = r_{n,p} H_{n-p}(t), \quad r_{n,p} = 2^p \frac{n!}{(n-p)!}. \tag{30}$$

The function $u(t) \in L^2_{\omega(t)}(\mathbb{R})$, with a suitable weight function $\omega(t) = e^{-t^2}$ may be expressed in terms of Hermite polynomials as follows

$$u(t) = \sum_{k=0}^m c_k H_k(t) \tag{31}$$

Where the coefficients c_n are given by

$$c_n = \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} u(t) H_n(t) \omega(t) dt, \quad n = 0, 1, \dots \tag{32}$$

In practice, only the first $(m + 1)$ –terms of Hermite polynomials are considered. Then we have

$$u_m(t) = \sum_n^m c_n H_n(t). \tag{33}$$

The main approximate formula of the fractional derivative is given in the following theorem.

Theorem Let $u(t)$ be approximated by Hermite polynomials as (33) and also suppose $q > 0$, then

$$D^q(u_m(t)) \cong \sum_{n=q}^m \left[n! \sum_{k=q}^{\ell} c_n B_{n,k}^{(q)} t^{n-2k-q} \right], \tag{34}$$

where $\ell = \frac{n-q}{2}$ and $B_{n,k}^{(q)}$ is given by

$$B_{n,k}^{(q)} = \frac{(-1)^k 2^{n-2k}}{(k)! (n-2k-q)!}$$

Proof: since the non linear operation we have

$$D^q(u_m(t)) = \sum_{n=0}^m c_n D^q(H_n(t)). \tag{35}$$

It is clear that $D^q H_n(t) = 0$, $n = 0, 1, \dots, q - 1$, $q > 0$ Therefore, for $n = q, q + 1, \dots, m$

Substituting equation (29) in (35) we have

$$D^q(u_m(t)) = \sum_{n=0}^m n! c_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^{n-2k}}{(k)! (n-2k)!} D^q t^{n-2k}.$$

$$D^q(u_m(t)) = \sum_{n=q}^m n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_n \frac{(-1)^k 2^{n-2k}}{(k)! (n-2k-q)!} t^{n-2k-q}. \tag{36}$$

In this section, we implement the Hermite method to solve numerically the non-linear Logistic, Riccati differential equation

6.1 Hermite method for solving logistic differential equation

In order to use the Hermite method we approximate $u(t)$ with $m = 2$ as

$$u(t) \approx \sum_{k=0}^2 c_k H_k(t) \tag{37}$$

substituting equations (33) , (34) in equation (1) we obtain

$$\sum_{n=q}^m n! \sum_{k=q}^{\ell} c_n B_{n,k}^{(q)} t^{n-2k-q} = \rho \sum_{n=0}^m c_n H_n(t) \left[1 - \sum_{n=0}^m c_n H_n(t) \right] \tag{38}$$

We now collocate Equation (38) at $(m + 1 - q)$ point $t_p, p = 0, 1, \dots, m - q$ as

$$\sum_{n=q}^m n! \sum_{k=q}^{\ell} c_n B_{n,k}^{(q)} t_p^{n-2k-q} = \rho \sum_{n=0}^m c_n H_n(t_p) \left[1 - \sum_{n=0}^m c_n H_n(t_p) \right] \tag{39}$$

For suitable collocation points we use roots of Hermite polynomial $H(t)$. In this case, the roots t_p of Hermite polynomial $H(t)$ are

$$t_0 = 0.707106781187, \quad t_1 = -0.707106781187,$$

Also, by substituting formula (37) in the initial conditions (2) we can find

$$\sum_{n=0}^2 g_n c_n = u_0 \tag{40}$$

Where $g_n = H_n(0)$ and is defined in (28)

Equation (39) with of the initial conditions (40), give $(2m + 2)$ of non-linear algebraic equations which can be solved using the Newton iteration method.

6.2 Hermite method for solving Riccati differential

In order to use the Hermite method we approximate $u(t)$ with $m = 2$ substituting equations (33) , (34) in equation (3) we obtain

$$\sum_{n=q}^m n! \sum_{k=q}^{\ell} c_n B_{n,k}^{(q)} t^{n-2k-q} = 1 - \left[\sum_{n=0}^m c_n H_n(t) \right]^2 \tag{41}$$

We now collocate Equation (41) at $(m + 1 - q)$ points $t_p, p = 0, 1, \dots, m - q$ as

$$\sum_{n=q}^m n! \sum_{k=q}^{\ell} c_n B_{n,k}^{(q)} t_p^{n-2k-q} = 1 - \left[\sum_{n=0}^m c_n H_n(t_p) \right]^2 \tag{42}$$

For suitable collocation points we use roots of Hermite polynomial $H(t)$. In this case, the roots t_p of Hermite polynomial $H(t)$ are

$$t_0 = 0.707106781187, \quad t_1 = -0.707106781187,$$

Also, by substituting formula (37) in the initial conditions (4) we can find

$$\sum_{n=0}^2 g_n c_n = u^0 \tag{43}$$

where $g_n = H_n(0)$ and is defined in (28)

Equation (42) with of the initial conditions (43), give $(2m + 2)$ of non-linear algebraic equations which can be solved using the Newton iteration method.

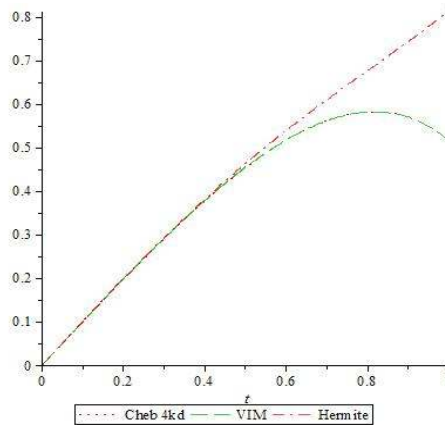


Fig. 4. The comparison between VIM, Chebyshev fourth kind Hermite polynomial

7 Conclusions

In this article, we used three computational methods, Chebyshev-spectral method, **VIM** and Hermite collocation methods to solve numerically the Logistic and Riccati differential equation. Using the Chebyshev-collocation method, the Logistic, Riccati differential equation reduced to a non-linear system of algebraic equations which solved by Newton iteration method. We presented a numerical simulation of the Logistic, Riccati differential equation with different values of m , the parameter ρ and the initial value u_0 . From the obtained numerical results, we can conclude that these three methods give us results in excellent agreement with the exact solution. In **VIM** it is evident that the overall errors can be made smaller by adding new terms from the sequence (12), but in Chebyshev-spectral method it is evident that the overall errors can be made smaller by adding new terms from the series. From our numerical results presented in all figures, it

is easy to conclude that the solution continuously depends on the initial condition and the value of the parameter ρ in the Logistic and Riccati differential equation.

Competing Interests

Authors have declared that no competing interests exist.

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