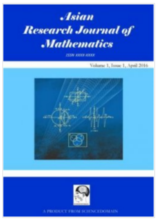

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Homfly Polynomial of Knotted Trivalent Plane Graphs

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Authors' contributions

This work was carried out in collaboration among all authors. All authors agreed on the idea of this paper. Author KQ managed the literature searches, put the plan of the study and typed the manuscript in Latex. Author AA also helped in putting the plan of the study and in managing the literature searches. Authors AA and EAE wrote some proof drafts and the first draft of the paper and revised the printed paper versions. Author ATD supervised the revision and alteration processes. All Authors read and approved the final manuscript.

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Abstract

We study the Homfly polynomial of periodic knotted trivalent plane graphs introduced in [1]. We show how periodicity of a knotted trivalent plane graphs is reflected in this polynomial. In particular, we derive congruences of periodic knotted trivalent plane graphs in terms of this polynomial invariant. These congruences yield criteria for periodicity of knotted trivalent plane graphs.

Keywords: Homfly polynomial; periodic links; knotted trivalent plane graphs.

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1 Introduction

Let p be an odd prime and $\mathbb{Z}_p$ be the finite cyclic group of order p. A knotted trivalent graph is said to be p-periodic of period p if there is an orientation preserving homeomorphism of $\mathbb{S}^3$ of order p, with a circle as a set of fixed points that is disjoint from the knotted trivalent planar graph G, that maps G onto itself setwise. By the positive resolution of the Smith conjecture [2], we can represent the diagram of a p-periodic knotted trivalent graph G as the closure of a p^{th} power of a knotted trivalent graph, where the product of two knotted trivalent graphs is obtained by connecting the top points of the second graph to the bottom points of the first one by a collection of parallel, non-weaving strands. This defines an action of the group $\mathbb{Z}_p$ on the knotted trivalent graph G by a $2\pi/p$-rotation.

Many congruences of periodic links have been given in terms of polynomial invariants such as the Jones polynomial [3] and its 2-variable generalization [4, 5, 6, 7, 8, 9, 10], the Alexander polynomial [11, 12, 13], and the twisted Alexander polynomials [14]. Different criteria of periodicity have been found in terms of hyperbolic structures on knot complements [15], homology groups of cyclic branched covers [16, 17], concordance invariants of Casson and Gordan [17], Khovanov homology [18], link Floer homology [19] and the Heegaard Floer correction terms of the finite cyclic branched covers of knots [20].

Our purpose in this paper is to explore periodicity of knotted trivalent plane graphs in terms of the Homfly polynomial invariant defined in [1]. The congruences of periodic knotted trivalent graphs are given in terms of this polynomial invariant. The restriction of these congruences to the case of links yields congruences of periodic links in terms of this polynomial.

2 The Homfly Polynomial of Knotted Trivalent Graphs

The authors of [1] defined an invariant of colored, oriented, trivalent, plane graphs that can be extended to an invariant of colored, oriented, trivalent, knotted graphs using the skein relations [1, Section 3].

We recall the definition of the above invariant and refer the reader to [1] for further details.

Our trivalent plane graphs are oriented such that at each vertex two edges are "in" and one edge is "out" or two edges are "out" and one edge is "in". The two in- or out-edges are called the legs of the vertex and the one out- or in-edge is called the head of the vertex.

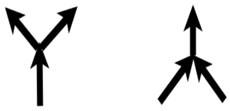
A coloring f of the graph is a map from the edge set to the set of positive integers less than or equal to $n \geq 2$ such that the sum of the colors of the legs is equal to the color of the head of that vertex.

A state σ is an assignment of a subset A of $\mathcal{N} = \{\frac{-n+1}{2}, \frac{-n+3}{2}, \ldots, \frac{n-1}{2}\}$ denoted by $\sigma(e)$ to each edge e such that $\#(A) = f(e)$ and the union of the subsets assigned to the legs coincides with that to its head, where $\#(A)$ is the number of elements of A and $f(e)$ is the integer assigned to the edge e from the given coloring f.

For any disjoint subsets A_1 and A_2 of $\mathcal{N}$ with $n \geq 2$, we put $\pi(A_1, A_2) = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\}$. The weight $wt(v;\sigma)$ of a vertex with a given state σ is defined as follows:

$$\mathrm{wt}(v;\sigma) = q^{\frac{f(e_1)f(e_2)}{4} - \frac{\pi(\sigma(e_1),\sigma(e_2))}{2}},$$

where q is an indeterminate, and e_1 and e_2 are left and right legs respectively with respect to the orientation of G (See the following figure).



Throughout this paper, diagrams that appear in one equation are identical except as indicated in a small disk.

Now if we replace each edge e by $f(e)$-parallel edges and assign to each copy an element of the subset determined by σ and connect at every vertex each pair of edges with the same element of $\mathcal{N}$, then we obtain a union of simple closed curves each of which equipped with the same element. Hence, the rotation number of the state σ denoted by $\mathrm{rot}(\sigma)$ is defined as follows:

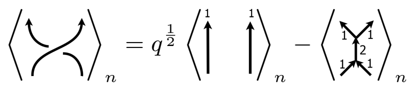
$$\mathrm{rot}(\sigma) = \sum_{C} \sigma(C)\,\mathrm{rot}(C),$$

where the sum is taken over all simple closed curves C equipped with $\sigma(C) \in \mathcal{N}$ and $\mathrm{rot}(C)$ is 1 if C is oriented counter-clockwise and -1 otherwise.

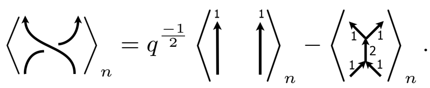
Now we define the mapping $\langle G\rangle_n$ to have value 1 for the empty graph and otherwise to be defined as follows:

$$\langle G\rangle_n = \sum_{\sigma:state} \{ \prod_{v:vertex} \mathrm{wt}(v;\sigma)\} q^{\mathrm{rot}(\sigma)}.$$

From definition, we conclude that this mapping is invariant under ambient isotopy of $\mathbb{R}^2$. Also, it can be extended to the case of knotted colored plane graphs using the following skein relations:

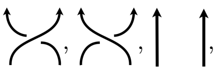

(2.1)

and


(2.2)

Now we define a new mapping in terms of this mapping $P_n(G) = q^{\frac{-nw(G)}{2}} \langle G\rangle_n$ for any trivalent knotted plane graph G, where $w(G)$ is the writhe of G. The last mapping is a specialization of the Homfly polynomial invariant defined for links in [21] as a result of [1, Theorem 3.2]. To be more precise, it satisfies a skein relation of trivalent knotted plane graphs as follows:

$$q^{\frac{n}{2}} P_n(G_+) - q^{\frac{-n}{2}} P_n(G_-) = (q^{\frac{1}{2}} - q^{\frac{-1}{2}}) P_n(G_0), \tag{2.3}$$

where G_+, G_- and G_0 are identical trivalent knotted plane graphs except near a crossing given by the following scheme: , respectively.

3 Main Results

As a result of Equations 2.1 and 2.2, it is easy to see that $P_n(G^{'})(q) = P_n(G)(q^{-1})$, where $G^{'}$ is the mirror image of the knotted trivalent plane graph G. For the next result, we define the ideal I of the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ generated by p and $q^p - 1$.

Theorem 3.1. *For any p-periodic knotted trivalent plane graph G, we have* $P_n(G) \equiv P_n(G_\nu) \mod I$, *where G_ν is the knotted trivalent graph G after interchanging all crossings of the orbit ν. Therefore, we conclude that* $P_n(G) \equiv P_n(G') \mod I$, *where G' is the mirror image of G.*

Proof. We apply Equations 2.1 and 2.2 to resolve all crossings of the orbit ν in G and the corresponding crossings in G_ν. By considering the state summations modulo p, we need only to examine the contributions of the graph states that are p-periodic. We can pair the terms in both summations in a way that the graph state in both terms is identical but with possibly different coefficients. It is clear that these coefficients are scalar multiple of each other by a factor of $\pm q^{\pm p}$. Therefore, we obtain $\langle G\rangle_n \equiv \langle G_\nu\rangle_n \mod I$ after applying the second relation of the ideal I. Now the result follows since $q^{\frac{-nw(G)}{2}} \equiv q^{\frac{-nw(G_\nu)}{2}} \mod I$ since the difference in writhes of G and G_ν is $\pm 2p$. Finally, we apply this to all orbits to obtain the second result. □

The next result deals with the p-periodic trivalent knotted graph G and its quotient G_*. For this purpose, we define a new ideal J of the ring $\mathbb{Z}[q^{\pm\frac{1}{2}}]$ that is generated by p and $[n]^p - [n]$, where $[n] = \frac{q^{\frac{n}{2}} - q^{\frac{-n}{2}}}{q^{\frac{1}{2}} - q^{\frac{-1}{2}}}$.

Theorem 3.2. *For any p-periodic knotted trivalent graph G, we have* $P_n(G) \equiv (P_n(G_*))^p \mod J$.

Proof. We use induction on the number of orbits of trivalent vertices in G. In the case G has zero orbits of trivalent vertices, then G represents a link diagram D of some link. Now there is a one-to-one correspondence between the binary resolving tree of D in computing $P_n(D)$ and the binary resolving tree of D_* in computing $P_n(D_*)$. Now Equation 2.3 implies the result if we use induction on the number of crossings and under the assumption that the result holds for crossing changes.

Now if G contains $(m+1)$ orbits of trivalent vertices, then with the aid of the formulas given in Equations 2.1 and 2.2, we obtain

$$G = q^{\frac{p}{2}} G_1 - G_2 \mod p,$$

for some p-periodic knotted plane trivalent graphs G_1, and G_2 of m pairs of trivalent vertices with quotient knotted trivalent graph

$$G_* = q^{\frac{1}{2}} G_{1*} - G_{2*},$$

where G_{1*} and G_{2*} are the quotient knotted trivalent graphs of G_1 and G_2 respectively. Finally, the result follows from the induction hypothesis on each term. □

The following results can be derived from the Theorem 3.2, but before we do so we need to analyze the second relation of the ideal J. To simplify notation, we let $u = q^{\frac{1}{2}}$. The second relation of the ideal J modulo p can be written in terms of u as follows:

$$\left(\sum_{i=0}^{[\frac{n}{2}]-1} (u^{n-(2i+1)} + u^{-n+(2i+1)})\right)^p - \sum_{i=0}^{[\frac{n}{2}]-1} (u^{n-(2i+1)} + u^{-n+(2i+1)}) =$$
$$\sum_{i=0}^{[\frac{n}{2}]-1} \Big((u^{n-(2i+1)} + u^{-n+(2i+1)})^p - (u^{n-(2i+1)} + u^{-n+(2i+1)})\Big)$$

Corollary 3.3. *For any p-periodic knotted trivalent graph G, we have*

1. *If n is odd, then* $P_n(G) \equiv (P_n(G_*))^p \mod (p, q^{2p} - q^{p+1} - q^{p-1} + 1)$

2. *If n is even, then* $P_n(G) \equiv (P_n(G_*))^p \mod (p, q^p - q^{\frac{p+1}{2}} - q^{\frac{p-1}{2}} + 1)$.

Proof. We prove the first case and the second case follows in the same manner. If n is odd, then all the powers in each term of the above sum are even. In particular, we have

$$\begin{aligned}(u^{2m} + u^{-2m})^p - (u^{2m} + u^{-2m}) &= (q^m + q^{-m})^p - (q^m + q^{-m}) \\ &= q^{pm} + q^{-pm} - q^m - q^{-m} \pmod p \\ &= q^{-pm}(q^{2pm} + 1 - q^{(p+1)m} - q^{(p-1)m}) \\ &= q^{-pm}(q^{(p+1)m}(q^{(p-1)m} - 1) - (q^{(p-1)m} - 1)) \\ &= q^{-pm}(q^{(p+1)m} - 1)(q^{(p-1)m} - 1).\end{aligned}$$

It is clear that the last polynomial is divisible by $(q^{p+1} - 1)(q^{p-1} - 1) = q^{2p} - q^{p+1} - q^{p-1} + 1$. The result follows since each of the above terms in the second relation of the ideal J is divisible by $q^{2p} - q^{p+1} - q^{p-1} + 1$. □

Corollary 3.4. *Let α and β denote a primitive $\left(\frac{p-1}{2}\right)$th-root of unity and $\left(\frac{p+1}{2}\right)$th-root of unity respectively, then*

1. $P_n(G)(\alpha) \equiv P_n(G_*)(\alpha) \pmod p$
2. $P_n(G)(\beta) \equiv P_n(G_*)(\beta^{-1}) \pmod p$

Proof. We prove the second case when n is odd and the other cases follow in the same manner. It is clear that $(q^{\frac{p+1}{2}} - 1)(q^{\frac{p-1}{2}} - 1)$ divides $q^{2p} - q^{p+1} - q^{p-1} + 1$. Therefore, the second relation of the ideal J is simply zero if $q = \beta$. Now from Theorem 3.2, we have $P_n(G)(\beta) \equiv (P_n(G_*)(\beta))^p \equiv P_n(G_*)(\beta^p) \equiv P_n(G_*)(\beta^{-1}) \pmod p$, since $\beta^p = (\beta^{\frac{p+1}{2}})^2\beta^{-1} = \beta^{-1}$. □

Corollary 3.5. *For any p-periodic knotted trivalent graph G, we have* $P_n(G) \equiv (P_n(G_*))^p \mod (p, q^{2n} - q^{n+1} - q^{n-1} + 1)$.

Proof. We can obtain the same result in Theorem 3.2 after replacing the second relation in the ideal J by $[n]^{p-1} - 1$ or equivalently $(q^{\frac{n}{2}} - q^{\frac{-n}{2}})^{p-1} - (q^{\frac{1}{2}} - q^{\frac{-1}{2}})^{p-1}$. Now we analyze this relation with $u = q^{\frac{1}{2}}$ and use the fact that $\binom{p-1}{j} \equiv (-1)^j \pmod p$.

$$\begin{aligned}(u^n - u^{-n})^{p-1} - (u - u^{-1})^{p-1} &= \sum_{j=0}^{p-1}(-1)^j u^{n(2j-p+1)} - \sum_{j=0}^{p-1}(-1)^j u^{(2j-p+1)} \\ &= \sum_{j=0}^{\frac{p-1}{2}}(-1)^j \left((u^{n(2j-p+1)} + u^{-n(2j-p+1)}) - (u^{(2j-p+1)} + u^{-(2j-p+1)})\right).\end{aligned}$$

Now each term of the above sum has the form $(u^{2mn} + u^{-2mn}) - (u^{2m} + u^{-2m})$ for some positive integer m. We have

$$\begin{aligned}(u^{2mn} + u^{-2mn}) - (u^{2m} + u^{-2m}) &= u^{-2mn}(u^{4mn} - u^{2mn+2m} - u^{2mn-2m} + 1) \\ &= q^{-mn}(q^{2mn} - q^{mn+m} - q^{mn-m} + 1) \\ &= q^{-mn}(q^{mn+m} - 1)(q^{mn-m} - 1)\end{aligned}$$

It is clear that the last expression is divisible by $(q^{n+1} - 1)(q^{n-1} - 1) = (q^{2n} - q^{n+1} - q^{n-1} + 1)$. Therefore, the sum is divisible by $(q^{2n} - q^{n+1} - q^{n-1} + 1)$ since each term in the above sum is divisible by this term. □

Corollary 3.6. *For any p-periodic knotted trivalent graph G with $p = n$, we have either* $P_n(G)(q) \equiv (P_n(G_*))(q) \mod (p, q^{2n} - q^{n+1} - q^{n-1} + 1)$ *or* $P_n(G)(q) \equiv (P_n(G_*))(q^{-1}) \mod (p, q^{2n} - q^{n+1} - q^{n-1} + 1)$.

Proof. The result follows since the second relation of the ideal implies that q is either a $(p-1)$th root of unity or $(p+1)$th root of unity. ☐

Remark 3.1. The criterion in Theorem 3.1 is valid for all positive integers $p \geq 2$ not necessarily for odd primes. All knotted trivalent plane graphs pass this criterion for $p = 2$ since $q = q^{-1}$ in this case.

As an application, we want to see how to apply the criterion in Theorem 3.1 to obstruct periodicity of an example of a knot.

Example 3.7. *From the table in [1], we have* $P_2(K) = -q^{\frac{9}{2}} + q^{\frac{5}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}}$, *where K is the trefoil knot while* $P_2(K^{'}) = -q^{\frac{-9}{2}} + q^{\frac{-5}{2}} + q^{\frac{-3}{2}} + q^{\frac{-1}{2}}$. *It is a simple exercise to show that* $P_2(K) \equiv P_2(K^{'}) \mod I$ *only if* $p = 3$. *Therefore, we can conclude that the trefoil is only 3-periodic.*

4 Conclusion

We show that the Homfly polynomial of periodic knotted trivalent plane graphs introduced in [1] satisfies some congruences. Therefore, the periodicity of a knotted trivalent plane graphs is reflected in this polynomial. From this, we derive criteria for periodicity of knotted trivalent plane graphs. For example, this leads to criteria for periodic links.

Competing Interests

Authors have declared that no competing interests exist.

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