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A New Quantum Paradox

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The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

A gauge transformation of a simple electromagnetic system is analyzed. The Hamiltonian which is derived from the Dirac Lagrangian density is used for determining the state of an electron. The fact that this Hamiltonian is free of time differential operators plays a key role in the analysis and proves that this Hamiltonian is *not* invariant under a general gauge transformation. An application of a specific gauge transformation illustrates this fact. These results call for a further analysis of the role of gauge transformations in the theoretical structure of electrodynamics.

Keywords: Lagrangian density; Dirac equation; Hamiltonian; Gauge transformations; Paradox.

1 INTRODUCTION

This work discusses a paradox that is obtained from an application of a gauge transformation to a simple electromagnetic system. Electrodynamics is a widely studied sector of theoretical physics and it is relevant to many physical disciplines, ranging from solid state physics to astrophysics. Therefore, the entire physical community is expected to belong the the readership of this work.

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A paradox is regarded as a useful tool for finding out new properties and interrelations between elements of a theory. A physical paradox describes a hypothetical device and relevant physical laws are assumed to determine the behavior of the system. The outcome of a paradox is an apparent contradiction. Such a contradiction provides a motivation for a further investigation of the relevant physical laws. This kind of investigation generally contributes to a better understanding of these laws. The following lines briefly describe two well known paradoxes which are used here as an illustration of this matter.

In the 1930s, Einstein, Podolsky and Rosen (EPR) described a quantum paradox of an action at a distance [1]. They used a principle which they called *physical reality* and regarded the result as an indication that quantum mechanics is an incomplete theory. For this reason, EPR put forward the need for finding hidden parameters that will promote quantum mechanics to the status of a complete theory. Later Bohm and Aharonov [2, 3] and Bell [4] have added elements that were used in an experimental test of the EPR idea. Experimental results support the idea that there is a kind of quantum information that propagates instantaneously (see [5] and references therein). Thus, the apparent EPR paradox has provided a motivation for acquiring new information on how physical processes work.

In the 1960s Shockley and James presented a paradox where a stationary system of a charge and a magnet has an electromagnetic nonzero linear momentum [6]. Soon after the publication of this paradox, Coleman and Van Vleck provided a general proof showing that the system's total linear momentum must be balanced [7]. Later Comay has shown that an explicit mechanical linear momentum exists in the system. In particular, if a nonvanishing pressure gradient exists along a closed loop of current then effects related to the energy-momentum tensor yield a nonzero mechanical linear momentum [8]. This mechanical momentum balances the electromagnetic linear momentum and also supports the validity of Coleman and Van Vleck general analysis. Thus, the Shockley and James paradox has ended up with a better understanding of elements of classical physics.

This paper discusses gauge transformations in the quantum domain. In classical physics, electro-dynamic equations of motion - namely, Maxwell equations and the Lorentz force - are independent of the 4-potentials. Therefore, classical electrodynamics is invariant under a gauge transformation. On the other hand, quantum theories depend explicitly on the 4-potentials. The analysis abides by physical laws and proves that the Dirac Hamiltonian is not invariant under a specific gauge transformation. This outcome demonstrates the need for a further analysis of the role of gauge in theoretical physics.

The paradox of this work is described in the second section. The third section contains the conclusions. Expressions are written in units where $\hbar = c = 1$. The relativistic metric is diagonal and its entries are (1,-1,-1,-1). Greek indices run from 0 to 3.

2 THE PARADOX

The paradox described below arises from an examination of a specific gauge transformation that pertains to the state of an electron which obeys the Dirac equation. To this end, let us examine the Lagrangian density of a Dirac electron [9, see p. 78]

$$\mathcal{L}_D = \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi, \qquad (2.1)$$

where $A^\mu = (V, \mathbf{A})$ denote the components of the electromagnetic 4-potential [10, see p. 10] or [11, see p. 48]. Here one sees that in this equation, like in any other quantum equation, the charge interacts with the 4-potential.

It is well known that the Lagrangian density of (2.1) is invariant under the gauge transformation $\Lambda(x)$ which is an arbitrary function of the space-time coordinates (denoted by x) [9, see p. 78] and [12, see p. 345]

$$A_\mu(x) \to A_\mu(x) + \Lambda(x)_{,\mu}\,;$$
$$\psi(x) \to \exp(-ie\Lambda(x))\psi(x). \qquad (2.2)$$

Here e is the electronic charge, which is a dimensionless Lorentz scalar in the units where $\hbar = c = 1$. Indeed, substituting (2.2) into (2.1),

one finds that the contribution of the gauge 4-potentials $\Lambda(x)_{,\mu}$ is canceled out by the additional terms obtained from the partial differentiation of $\exp(-ie\Lambda(x))\psi(x)$.

The symbol $\psi(x)$ of the Dirac Lagrangian density $(\,2.1)$ describes a general state of the given Dirac particle, simply because this Lagrangian density holds for all cases. The purpose of the following discussion is to find out how a gauge transformation affects *specific solutions* of the Dirac equations. To this end, one must construct the Hamiltonian and pick up the required solution from the entire spectrum of its eigenfunctions. The discussion presented below is dedicated to this matter.

Let us turn to the paradox and examine a motionless electron located at the vicinity of point P in a field-free space. The Dirac Hamiltonian is used for finding the time evolution of this electron. (A quantum expression for the Hamiltonian is also required by the Bohr correspondence principle. Here the classical limit of quantum theories should agree with classical physics. Evidently, in classical physics energy is a well defined quantity. Therefore, one requires that quantum theories should have a self-consistent expression for energy.) This Hamiltonian can be derived from the Lagrangian density of (2.1) in the following steps.

The Hamiltonian density $\mathcal{H}$ is derived from the Lagrangian density by the well known Legendre transformation

$$\mathcal{H}=\sum_i \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}\dot{\psi}_i-\mathcal{L}, \tag{2.3}$$

where the index i runs on all functions. In the specific case of a Dirac particle one obtains from (2.1) and (2.3)

$$\mathcal{H}_D=\psi^\dagger[\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})+\beta m+eV]\psi, \tag{2.4}$$

which is written here in the standard notation [10, see p. 11]. The density of a Dirac particle is $\psi^\dagger\psi$ [10, see p. 9]. Thus, removing the density from (2.4), one obtains the operator form of the Dirac Hamiltonian

$$H_D=[\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})+\beta m+eV]. \tag{2.5}$$

This Hamiltonian stands on the right hand side of the Dirac equation [10, see p. 11]

$$i\frac{\partial\psi}{\partial t}=H_D\psi=[\boldsymbol{\alpha}\cdot(-i\nabla-e\mathbf{A})+\beta m+eV]\psi. \tag{2.6}$$

As is well known, the Dirac Hamiltonian $(\,2.5)$ does not contain a time differential operator.

The Dirac equation (2.6) is used for finding the time evolution of an electron at the vicinity of point P. Here the field-free 4-potential is

$$A^\mu=0. \tag{2.7}$$

Hence, the Dirac equation for a free electron

$$i\frac{\partial\psi}{\partial t}=[\boldsymbol{\alpha}\cdot(-i\nabla)+\beta m]\psi \tag{2.8}$$

determines the electronic state.

Let us examine how this system is affected by the following gauge function

$$\Lambda(x)=et/r. \tag{2.9}$$

Here e is the absolute value of the electronic charge, t is the time and r is the distance from the origin of the spatial coordinates. Certainly, the gauge function of (2.9) is a legitimate gauge expression because it depends on space-time coordinates. This gauge transformation casts the null 4-potential of $(\,2.7)$ into the following expression

$$A'_\mu=\frac{\partial(et/r)}{\partial x^\mu}=(e/r,-et\mathbf{r}/r^3). \tag{2.10}$$

The gauge transformation of (2.2) also transforms the Dirac wave function. Introducing the specific gauge function of $(\,2.9)$, one finds that the transformed wave function (2.2) is

$$\psi'(x)=\exp(ie^2t/r)\psi(x). \tag{2.11}$$

And indeed, substituting (2.10) and (2.11) into the Dirac Lagrangian density $(\,2.1)$, one finds that this Lagrangian density is invariant under the transformation of the gauge function (2.9).

On the other hand, the new function $\psi'(x)$ of $(\,2.11)$ must satisfy the Dirac equation $(\,2.6)$, where the gauge terms of (2.10) are used in the

expression for the 4-potential. Here one obtains

$$\begin{aligned} i\frac{\partial\psi'}{\partial t} &= H_D\psi'(x) \\ &= [\boldsymbol{\alpha}\cdot(-i\nabla - e\mathbf{A}) + \beta m + eV]\psi'(x) \\ &= \exp(ie^2t/r)[\boldsymbol{\alpha}\cdot(-i\nabla) + \beta m - e^2/r]\psi(x) \end{aligned} \tag{2.12}$$

It turns out that similarly to the case of the Dirac Lagrangian density (2.1), the contribution of the 3-vector part of the gauge (2.10) is eliminated from the Dirac Hamiltonian of (2.12). On the other hand, the 0-component of that gauge remains as is. This outcome stems from the fact that the Dirac Hamiltonian (2.5) contains spatial differential operators but is free of a time differential operator.

Let us define the point P of the electron so that its distance from the origin of the spatial coordinates is about the Bohr radius. It turns out that the gauge transformation (2.9) produces (2.12) and the differential equation inside the square brackets of its last line is the Dirac equation of a bound electron of the hydrogen atom [10, see p. 52], because it contains an additional Coulomb-like term $-e^2/r$. This equation is an eigenvalue problem which is thoroughly discussed in the literature. In particular, the bound states of the hydrogen atom have a radially decreasing exponential factor of the form $\exp(-kr)$, where k is a positive constant [10, see p. 55]. Now, $\psi'(x)$ and $\psi(x)$ of (2.12) differ by a phase factor whose absolute value is unity. It means that also the absolute value of $\psi'(x)$ decreases exponentially. Evidently, the exponentially decreasing factor of the solution $\psi'(x)$ of the gauge transformed problem of (2.9) is inconsistent with the free wave of the electronic state of the null potential (2.7). This result proves that an application of the legitimate gauge transformation (2.9) yields a paradox.

The foregoing analysis shows a problem with the gauge-transformed Dirac Hamiltonian which stands on the right hand side of (2.12), because it contains the unphysical Coulomb-like term $-e^2/r$. This outcome indicates that a corresponding problem should exist with the left hand side of this equation. And indeed, it is proved here that this additional unphysical term is also found on the left hand side of (2.12). Thus, let us examine the gauge transformed function ψ' of (2.11) which stands on the left hand side of (2.12). As is well known, an ordinary wave function of a motionless particle in a well-defined energy state takes the form

$$\psi(x) = \exp(-iEt)\chi(x,y,z). \tag{2.13}$$

Here the time dependence of $\psi(x)$ appears only in the phase where the energy E is a constant and $\chi(x,y,z)$ is a spatially dependent energy eigenfunction. On the other hand, the phase factor of the gauge transformed function ψ' of (2.11) also depends on the radial coordinate r and on the time t. In the present case one obtains for the motionless free electron

$$\begin{aligned} i\frac{\partial\psi'}{\partial t} &= i\frac{\partial\exp(ie^2t/r)}{\partial t}\psi + i\exp(ie^2t/r)\frac{\partial\psi}{\partial t} \\ &= (-e^2/r + m)\psi', \end{aligned} \tag{2.14}$$

where m is the electronic mass. Therefore, the coordinate-dependent quantity $-e^2/r$ of (2.14) proves that the gauge-transformed wave function ψ' is *not* an energy eigenfunction. This is a contradiction because an electron in a free space has a well defined energy and in the case of a motionless electron $E = m$ [10, see p. 28]. Hence, the same contradiction appears on each side of (2.12) where the unphysical Coulomb-like term $-e^2/r$ appears. This analysis shows a counter-example which proves that the gauge phase factor of the wave function ψ' of (2.12) destroys a specific eigenfunction of the Hamiltonian and casts it into an unacceptable form.

The foregoing discussion shows the two sides of the inconsistency that emerges from the application the gauge transformation (2.9) to the quite simple state of a motionless Dirac particle in a field free space. The transformed Hamiltonian (2.12) has a new unphysical term $-e^2/r$ that takes the form of the hydrogen atom problem. Hence, its eigenfunctions should be those of the hydrogen atom. On the other hand, (2.14) shows that the same gauge transformation casts an eigenfunction of a motionless particle into a function that is not an eigenfunction of the operator $i\partial/\partial t$.

It is interesting to note that the function ψ is included in the Dirac Lagrangian density (2.1) and

in the Dirac Hamiltonian density (2.4). Therefore, the problem with the gauge transformed function ψ' which is proved in (2.14) applies to the Lagrangian formalism and to the Hamiltonian formalism as well.

3 CONCLUSIONS

This work relies on three general principles of quantum theories of electromagnetic systems: the variational principle and its Lagrangian density, the key role of the Hamiltonian which is derived from this Lagrangian density and the gauge invariance of the system. Another general principle is the reliability of mathematical results that are obtained from an analysis of fundamental mathematical expressions of a physical theory.

The main result of this work is derived from an application of these principles. The analysis proves the following new property of the Hamiltonian of an electrically charged Dirac particle and of the associated function ψ.

- Unlike the Lagrangian density, which is invariant under a gauge transformation, *the associated Hamiltonian is not invariant under such a transformation.*
- The gauge transformation casts the function ψ into a physically unacceptable form. This point applies to the Lagrangian formalism and to the Hamiltonian formalism as well.

The first conclusion depends on the well known fact that the Dirac Hamiltonian is free of a time-derivative operator.

The paradox described herein provides an illustration of this conclusion. This paradox is related to two expressions of the 4-potential (2.7) and (2.10), which differ by a gauge transformation. It turns out that contrary to a general expectation, the Dirac Hamiltonian yields two physically different results. The state of a free electron which is derived from (2.8) is inconsistent with that of the solution of (2.12), where the electron is bound to the hydrogen atom and its wave function decreases exponentially with the distance r from the origin. This outcome illustrates the main point of this work: the Hamiltonian is *not* invariant under a general gauge transformation. Obviously, the Hamiltonian is a crucial element of classical and quantum theories because it determines the time evolution of the system. Furthermore, a corresponding paradox is found in the gauge transformed function ψ' (see (2.14)) which takes a physically unacceptable form. Other problematic aspects of the gauge transformations have been published earlier [13].

The problematic aspects of the result of this work apply to quantum mechanics and to quantum field theory (QFT) as well. Indeed, the close relationships between these theories is stated clearly in S. Weinberg's well known textbook: "First, some good news: quantum field theory is based on the same quantum mechanics that was invented by Schroedinger, Heisenberg, Pauli, Born, and others in 1925-26, and has been used ever since in atomic, molecular, nuclear and condensed matter physics" (see [12], p. 49). The same conclusion can also be found in Rohrlich's textbook which explains the hierarchical relationships between QFT and quantum mechanics (see [14], pp. 1-6).

The result of this work provides a motivation for a further investigation of the role of gauge in electrodynamics and of the Lagrangian-Hamiltonian relationships. Such an investigation is expected to end up with a deeper understanding of the role of gauge in electrodynamics.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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