British Journal of Education, Society & Behavioural Science
16(1): 1-13, 2016, Article no.BJESBS.26207
ISSN: 2278-0998

SCIENCEDOMAIN *international*
www.sciencedomain.org

Solving Non-linear Equations Systems through Numerical Methods

Costică Lupu[1*]

[1]Department of Mathematics-Informatics and Education Sciences, Faculty of Sciences, "Vasile Alecsandri" University of Bacău, Str. Mărășești 157, 600115, România.

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJESBS/2016/26207
Editor(s):
(1) Shao-I Chiu, Taipei College of Maritime Technology of Center for General Education, Taiwan.
Reviewers:
(1) Elakkad Abdeslam, CRMEF Fès, Morocco.
(2) Belal Sadiq Hamed Rabab'h, Ministry of Education, Jordan.
(3) Krasimir Yordzhev, South-West University, Bulgaria.
Complete Peer review History: http://sciencedomain.org/review-history/14574

Original Research Article

Received 5th March 2016
Accepted 4th May 2016
Published 11th May 2016

ABSTRACT

This study investigate the contribution assessment methods for solving linear systems of equations and graphical functions by numerical methods to improve school performance. Learning becomes effective only when students actively participate in the learning process: discussions, arguments, investigations, experiments, etc., become sine qua non methods for effective and lasting learning. All the situations, not just the active methods themselves, with which the students are faced and that remove them from their hypostasis of object of training, turning them into active subjects of education involved in their training, are forms of active learning.

The experimental study was based on the following *hypothesis*: The researcher use traditional teaching-learning strategies combined with the appropriate modern ones, then students will choose to consciously and effectively acquire the methodology for solving exercises and problems, reinforcing the idea that Mathematics, a universe of knowledge and an educational subject, may be understood in a smart, methodical manner, thus facilitating the progressive closeness to her beauty by reasoning and related calculation techniques, disciplinary effort, the logic of intelligently suggested and discovered connections.

Based on this hypothesis, there were established the research objectives, namely: - knowledge of

**Corresponding author: E-mail: costica_lupu@yahoo.com;*

the students' psychological characteristics and determining their baseline levels; - correct and conscious assimilation of the concepts taught; - building problem-solving and investigation skills; - the optimal integration of evaluation processes during lessons of Mathematics by using effective work tools and techniques; - achieving optimal synchronization between learning and recognition of the learning outcomes, focusing on increasing motivation for learning.
The experiment was conducted at two high-schools, involving three 11th grades from the Pedagogical College "Ștefan cel Mare" from Bacău (60 students) and three 11th grades from the National College "Vasile Alecsandri" from Bacău, (60 students), in the 2014-2015 school year.
Calculating the mean of the two tests (initial and final) and drawing a comparison between the two groups, there may be noticed an increase in the school performance at the experimental group as compared to the control group.
The results obtained by the students confirm the research hypothesis.
In conclusion, the researcher recommends: to achieve high-quality education and obtain the best results, we should combine classical and modern teaching, learning and evaluation methods, different forms of organization and a variety of teaching tools and materials.
In this article we will focus on non-linear systems of equations. The topic is interdisciplinary. It is a combination of two sciences - mathematics and programming. The work is primarily methodological and it is intended for teachers and students.

Keywords: Numerical methods; non-linear systems; graphical representation; formative-ameliorative research.

1. INTRODUCTION

The problem discussed in this chapter can be generically written as f(x) =0, (1) but admits different interpretations, depending on the meaning of x and f. Equation with one unknown, a case in which f is a function given by a real or complex variable and we try to find the values of this variable for which f cancelled. Such values are called the roots of the equation (1) or the zeros of the function f. If x in (1) is a vector x = $[x_1, x_2, . . ., x_d]$ and f is also a vector whose components are functions of the variables x_1, x_2, . . . , x_d , then (1) is a system of equations.

It is said that the system is non-linear if at least one of the components of f depends, in a non-linear way, on at least one of the variables $x_1,x_2,. . .,x_d$. If all the components of *f* are linear functions de x_1,x_2 , . . . , x_d, then the system is a system of linear algebraic equations.

In more general terms (1) could be a functional equation if x is an element of a function space and f is an operator (linear or nonlinear) acting on this space. In each of these situations, the zero from the right side of (1) can have various interpretations: the number zero in the first case, the null vector in the second and the identically null function in the third case. Much of this chapter is devoted to a nonlinear scaling. Such equations occur frequently in vibration analysis systems, where the roots correspond to critical frequencies (resonance) [1].

This article we will focus on non-linear systems of equations. To solve these systems we shall use either the method of substitution or the removal method that we have analysed when solving systems we could reach complex solutions, in addition to real solutions. Just as we saw in the solving systems of two equations, the real solutions will represent the coordinates for the points where the two graphs of the functions intersect [2].

2. THEORETICAL ASPECTS

2.1 The Theoretical Foundation

Let there be the system of equations [3]:

$$\begin{cases} f_1(x_1, x_2, ..., x_n) = 0 \\ f_2(x_1, x_2, ..., x_n) = 0 \\ \\ f_n(x_1, x_2, ..., x_n) = 0 \end{cases} \quad (2)$$

where the functions f_k:A$\subset$ R^n, A *open*, $f_k \in$ C^1(A), and the Jacobean.

$$\frac{D(f_1,f_2,\dots,f_n)}{D(x_1,x_2,\dots,x_n)}(x)=\begin{vmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \dots & \frac{\partial f_2}{\partial x_n}(x) \\ & \dots & & \\ \frac{\partial f_n}{\partial x_1}(x) & \frac{\partial f_n}{\partial x_2}(x) & \dots & \frac{\partial f_n}{\partial x_n}(x) \end{vmatrix} \neq 0 \quad (\forall)\ x=(x_1, x_2, \dots, x_n)\in A \qquad (3).$$

The solution $\mathbf{y=(y_1, y_2, \dots, y_n)\in A}$ of system **(1)** is determined with the help of the series of successive approximations, as shown below:

- The first approximation of the solution y is (arbitrarily) selected: $x^{(0)}=(x^{(0)}_1, x^{(0)}_2, \dots, x^{(0)}_n)\in A$; (4)
- The are successively determined the approximations of the solution y:

$$[x^{(1)}] = [x^{(0)}]-[J_f\, x^{(0)}]^{-1}\cdot[f(x^{(0)})] \in A\ ;\ [x^{(2)}] = [x^{(1)}]-[J_f\, x^{(1)}]^{-1}\cdot[f(x^{(1)})] \in A\ , \dots ,$$
$$[x^{(m)}] = [x^{(m-1)}]-[J_f\, x^{(m-1)}]^{-1}\cdot[f(x^{(m-1)})] \in A; \qquad (5)$$

where,

$$\left[x^{(k)}\right]=\begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{bmatrix} \qquad (6)$$

$$\left[f(x^{(k)})\right]=\begin{bmatrix} f_1(x_1^k,x_2^k,\dots,x_n^k) \\ f_2(x_1^k,x_2^k,\dots,x_n^k) \\ \dots \\ f_n(x_1^k,x_2^k,\dots,x_n^k) \end{bmatrix} \qquad (7)$$

$$\left[J_f(x^{(k)})\right]=\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_1^k,x_2^k,\dots,x_n^k) & \frac{\partial f_1}{\partial x_2}(x_1^k,x_2^k,\dots,x_n^k) & \dots & \frac{\partial f_1}{\partial x_n}(x_1^k,x_2^k,\dots,x_n^k) \\ \frac{\partial f_2}{\partial x_1}(x_1^k,x_2^k,\dots,x_n^k) & \frac{\partial f_2}{\partial x_2}(x_1^k,x_2^k,\dots,x_n^k) & \dots & \frac{\partial f_2}{\partial x_n}(x_1^k,x_2^k,\dots,x_n^k) \\ & & \dots & \\ \frac{\partial f_n}{\partial x_1}(x_1^k,x_2^k,\dots,x_n^k) & \frac{\partial f_n}{\partial x_2}(x_1^k,x_2^k,\dots,x_n^k) & \dots & \frac{\partial f_n}{\partial x_n}(x_1^k,x_2^k,\dots,x_n^k) \end{bmatrix} \qquad (8)$$

where k=0,1,2,...,m,... [4].

The approximation $y=x^{(m)}$ is considered satisfying when

$$\left\|x^{(m)}-x^{(m-1)}\right\|=\sqrt{\sum_{k=1}^{m}\left(x_k^m-x_k^{m-1}\right)^2}<\varepsilon, \qquad (9)$$

where **ε>0**, however sufficiently small, represents the precision of the solution, initially assigned [5].

2.2 Program Presentation

The program Sistem Neliniar uses two sub-programs: the function FUNCTIE and the procedure INVERSARE. The main program performs the following functions:

- Provides interactive introduction of the values of input variables (the space size) R^n**,** $n \in N^*$, the imposed accuracy **eps**, the number of allowed iterations nmax, the increase needed for the numerical evaluation of the partial derivatives **h**, the elements of the first approximation (3), x1[i]**,** i$\in \overline{1,n}$;

Carries out the processing corresponding to the evaluation of successive approximations (4) of the solution for the system (1), respectively:

- Calculates the vector $[f(x^{(k)})]$, according to no. (6);
- Calculates the matrix $[J_f(x^{(k)})]$, according to no. (7), numerically evaluating the corresponding derivatives;
- Calculates, with the help of the procedure INVERSARE, the inverse matrix $[J_f(x^{(k)})]^{-1}$;
- Evaluates the distance (8) between two successive approximations;
- Repeats all the previous processing, until achieving the desired precision or the number of iterations allowed, nmax;
- Displays the last solution approximation, the distance between the last two successive approximations and the number of iterations performed [6].

The function FUNCTIE calculates the values of the functions f_i**,** i$\in \overline{1,n}$**,** in the current point $x=(x_1,x_2,...,x_n)$**.** The parameters n and nfunc specify the size of space $\boldsymbol{R^n}$ and, respectively, the position i of the function f_i in the vector [f(x)]**.** The sub-program FUNCȚIE uses, besides the standard functions from the language (Borland) PASCAL, a series of functions defined within it that may be required to develop the functions f_i, i $\in \overline{1,n}$, respectively:

- The function PUTERE (e,v) that evaluates the function v^e, the variable being v;
- The function EXPA(e,v), that evaluates the function e^v, the variable being v;
- The hyperbolic functions sh(v) and ch(v), that evaluate the functions shv and chv;
- The functions asin(v) and acos(v), that evaluate the functions arcsin v and, respectively, arccos v.

The alternatives for the statement CASE are completed with the expressions of the functions $f_i(x_1,x_2,...,x_n)$**,** thus the program may be used for various systems of equations (1) to be solved.

The procedure Inverse performs the calculus of the inverse Jacobean matrix $[J_f(x^{(k)})]^{-1}$, in point $x^{(k)}=(x^k_1, ...,x^k_n)$ [7].

The source program

```
PROGRAM SISTEM_NELINIAR;
CONST Dim = 10;
TYPE
...vector = array[1..Dim] OF real;
...matrice = array[1..Dim,1..Dim] OF real;
VAR
...x,x1,x2,f: vector;
...a,b: matrice;
...n,nmax,i,j,k: integer;
...eps,h,suma,dist: real;

FUNCTION FUNCTIE(n,nfunc: integer; x: vector): real;
{ calculates the point value of the function as x }
CONST alfa = 1E-20;

FUNCTION PUTERE(e,v:real):real;
{function 'v power e', argument v}
VAR t: real;
BEGIN
...t := e*ln(v); PUTERE := exp(t)
END;{PUTERE}
```

Important. Filling in the corpus of the function FUNCTIE

```
BEGIN {corpus function FUNCTIE}
CASE afunc OF
...1: FUNCTIE:=x[2]*sin(x[1])+cos(x[2])- exp(x[3]);
...2: FUNCTIE:=exp(x[1])-sqr(x[2])+x[3]);
...3: FUNCTIE:=x[1]*sqr(x[1])+sqr(x[2])-
...sqr(x[3])*exp(x[3])
END
END;{FUNCTIE}
```

2.3 Application

Solve the system of equations (10) using the program SISTEM_NON-LINIAR [8].

$$\begin{cases} y\sin x+\cos y-e^{z}=0 \\ e^{x}-y^{2}+z=0 \\ x^{3}+y^{2}-z^{2}e^{y}=0 \end{cases} \tag{10}$$

Using the identifications: $x \equiv x[1]$, $y \equiv y[2]$, $z \equiv z[3$,the statement CASE from the function FUNCTIE becomes:

```
CASE nfunc OF
...1: FUNCTIE:= x[2]*sin(x[1])+cos(x[2])-exp(x[3]);
...2: FUNCTIE:= exp(x[1])-sqr(x[2])+x[3];
...3: FUNCTIE :=x[1]*sqr(x[1])+sqr(x[2])-sqr(x[3])*exp(x[2]);
END
```

Note. Using as the first approximation the vector $x^{(0)}=(1,1,2)$**,** where the circumstances under which the imposed precision was $\varepsilon=10^{-6}$, the increase of **h** for the numerical derivation was $h=10^{-4}$ and the number of allowed iterations was nmax=30, the approximate solution was obtained after 11 iterations:

x=-0,31628
y=0,56810
z=-0,40612.

The results of running the program are [9]:

system dimension: n=...3
desired precision: epsilon=...1E-6
no. of allowed approximations: nmax=...30
the increase needed for the numerical evaluation of the derivatives =...1E-4
first approximation:
...x(1)=1
...x(2)=1
...x(3)=2
solution:
...x(1) = -0.31628
...x(2) = 0.56810
...x(3) = -0.40612
distance between the last two approximations:
||xn-xn-1|| = 3.8427792511E-10
there were required 11 approximations

3. MATERIALS AND METHODS

3.1 The Research Methodology

The research relied on the following knowledge methods and techniques:

1. *The method of observation* that is frequently used in school. Both spontaneous (passive) and scientific (generated) observation support the accumulation of a rich factual material, being able to provide data on the students'

behaviour during lessons, breaks, extra-curricular and family activities.

2. *The method of conversation* was used to gather information from students, parents, the family's general practitioner or other acquaintances of the students. Thus, we could access data on the students' interests and aspirations, temperamental particularities, character features, general intelligence, family climate, material circumstances, daily regime, health, hobbies, likes/ dislikes in relation to certain activities, possibilities for doing homework.
3. *The psychological analysis of the activity's results/ products* provides information on several aspects related to the products of the activity. The data collected through this method was analysed by detaching appreciations and estimations related to the students' individuality, behaviour, inclinations and interests, the way in which they do their homework, their concern for correctness.
4. *The method of tests* represents a method that supports the diagnosis of the subject's development level – in this case, students – and formulating, on this basis, a prognosis regarding their evolution. *Docimological tests* provide quantitative information on the investigated phenomenon, when applied regularly during the instructive-educational process from the classes of Mathematics, as well as from other disciplines, have supported the determination of the level of knowledge, skills and the level of development of intellectual skills. They were conceived in relation with the established operational objectives, comprising sets of items meant to help us record and evaluate school performances.
5. *The statistical-mathematical methods* were used to analyse the obtained results that were inserted in analytical and synthetic tables, then systematized in centralized tables, graphs, histograms, circular diagrams, supporting the interpretation of data [10].

3.2 Research Description

The 60 students in the experimental sample were prepared of the 30 students practicing during the 48 lessons, supported the teaching practice of school second semester 2014/2015 in IT laboratories and in math. Students with the 3 mentors used in teaching and learning mathematics such content, active participatory methods modern. The other 60 students from control samples and three mentors continued teaching with traditional methods.

The paper highlights the role and values of computer use in learning Mathematics, in the informative as well as formative-educational sense, in agreement with the taught objectives and contents, based on the tendencies of updating and upgrading the school activity, and enhancing its role in preparing the student for life.

3.3 Sample Description

3.3.1 Sample description

To demonstrate the hypothesis, the experiment was conducted at two high-schools, involving three 11^{th} grades classes from each of them:

- The classes XI A, XI B, XI C – the experimental sample from the Pedagogical College "Ștefan cel Mare" from Bacău (60 students);
- The classes XI A, XI B, XI C – the control/ witness sample from the National College "Vasile Alecsandri" from Bacău, (60 students), in the 2014-2015 school year.

The classes of students included in the experiment are social groups formed during high-school. The 120 students involved in the research are homogeneous in terms of age, school level and nationality. Most of them come from families interested in school, knowledge and a good partnership with the school and teachers.

Nevertheless, the intellectual and socio-emotional experiences provided by these classes of students constituted the balance of a beautiful collaboration, where the insufficient knowledge from the initial point was compensated by their constant interest, curiosity, cooperation and perseverance.

3.4 Research Objectives

Modern times, social progress and educational reform, with their implications in modernizing the instructive-educational process requires that teachers complete their profile of knowledge assimilator and transmitter with that of investigator of educational phenomena and, based on this, of creator of ideas who enriches the psycho pedagogical theoretical patrimony and improve the educational structure.

To this end, I have set the following objectives for an experiment conducted during lessons of Mathematics:

O_1 – enhancing the learning motivation and, implicitly, school performance, as a result of solving exercises and problems using computers and numerical methods during classes of Mathematics;

O_2 – building independent work skills and independent thinking by the variety of the situations with which the student is faced;

O_3 – building basic scientific notions and modern creative mathematical thinking (swift, flexible thinking, high transfer capacity);

O_4 – monitoring and recording the students' progress during and at the end of the applicative research;

O_5 – the practical exploitation of the discussed and analysed results obtained after solving problems in class as well as in IT laboratories.

3.5 Research Hypothesis

In the case of the research conducted, researcher established the following hypothesis: Suppose that the most efficient strategies are used in the activity of learning arithmetical operations, then the results obtained will lead to enhanced school performance and school success.

3.6 Research Variables

The research hypothesis generates two research variables: - the independent variable, introduced through the numerical methods for solving systems and the graphical representation of functions; - the dependent variable related to enhancing the motivation for acquiring mathematical notions and school progress.

3.7 Research Stages

This theme has been under research during the school year 2014 - 2015, at two schools simultaneously: the Pedagogical College "Stefan cel Mare" from Bacău (60 students) and the National College "Vasile Alecsandri" of Bacău (60 students). During this period, there were applied tests that included the use of traditional methods combined with modern one in Math classes.

4. RESULTS AND DISCUSSION

4.1 Interpretation of the Initial Evaluation Results

To measure and assess the degree of achievement of the objectives proposed in this research, the students were applied an initial evaluation test. For data centralization and interpretation we have resorted to analytical and synthetic tables, frequency polygons, histograms. The results obtained after the application of the first test (initial evaluation) were noted in summary tables.

4.1.1 Experimental group

The analysis of the analytical and synthetic tables, the histogram, the frequency polygon and the circular diagram has revealed the following results for the experimental group in the initial evaluation: of the 60 evaluated students, 43% of them obtained the mark VW (very well), 36% of the participants obtained the mark W (well), and 21% of the participants obtained the mark S (sufficient).

4.1.2 Control group

The analysis of the analytical and synthetic tables, histogram, frequency polygon and circular diagram has revealed the following results of the control group in the initial evaluation: of the 60 evaluated students, 43% of them obtained the mark VW (very well), 30% of the participants obtained the mark W (well), 16% of the participants obtained the mark S (sufficient) and 11% obtained the mark I (insufficient).

4.2 Interpretation of the Formative Evaluation Results

The formative evaluation tests applied during the lessons of Mathematics enabled the immediate identification of the students' learning difficulties. In order to eliminate errors, the activity was differentiated and computers were used in solving systems of equations and in the graphical representation of functions. Following the analysis of the tests, there were revealed the operational objectives that were not achieved by students, so that they could be pursued in the proposed recovery activities.

Analyzing the data, we may say that the students from the experimental group did not record any relevant progress in terms of their marks, but it can be observed that all the students got better scores that in the previous test and the success of learning was ensured. Also, it was observed that the most frequent mistakes were calculus errors, denoting the fact that the methods used in heuristic problem solving are known and acquired by the students of the class.

The formative evaluation tests applied at lessons of Mathematics allowed for the immediate identification of the students' mistakes and learning difficulties. Comparing the tables and graphs containing the scores and marks for the ameliorative formative tests with those for the initial test, there may be observed an increase in the school performance, numerically expressed as follows: - the arithmetic mean in the initial test for the experimental class was 7,3 whereas in the formative test no. 1 it was 8; - there was a slight growth in the formative test no. 2 compared to the first test, the obtained arithmetic mean being 8,2.

This growth is the result of surpassing the relevant difficulties related to the contents of the learning unit on solving systems of equations and the graphical representation through numerical methods, involving computer use and informatics knowledge. The scores obtained were much higher than in the previous test. The obtained results highlight the relevance of formative tests applied during the learning activities as they confirm the usefulness of the applied heuristic methods and modern techniques. The fact that the results of the students from the experimental class recorded growth, even those students who did not learn systematically having reached the promotion level, determined us to intervene with recovery worksheets in order to repeat certain tasks and achieve better consolidation.

The progress obtained by the students as compared to the initial test cannot be accounted for only by raising the progress rate in achieving objectives, but also by the usefulness of the computer in solving the systems through numerical methods which activated or raised the desire for performance and, implicitly, a much more active and conscious participation of the students.

4.3 Interpretation of the Final Evaluation Results

The final evaluation test of the students was applied on June 1st 2015. For data centralization and interpretation we have resorted to analytical and synthetic tables, frequency polygons, histograms. The final evaluation test was conceived in a manner similar to the initial one, so that the results may be compared, the knowledge established by the syllabus being defined as operational objectives encoded as items.

Final Assessment Test

(3 points) is considered $n \in N^*$ and matrix $A_n \in M_n(R)$, which is on the main diagonal elements equal to 2 and the remaining elements equal to 1.

(1 point) Compute det. $(2A_2)$;
(1 point) Determine $x \in R$ for which det $(A_2 + xI_2)=0$;
(1 point) Show that the A_4 is reversed;
(2 point) If B_4^{-1}, to be resolved by numerical system $X \cdot B=C$, where $X=(x_1\ x_2\ x_3\ x_4)$ and C= (1 2 3 4) [11].

It is believed the function $f: R \to R$, $f(x) = \frac{2x^3}{x^2+1}$

(1 point) Show that the graph of the function f "admits asymptote at + ∞";
(1 point) Show that the function f is reversed;
(1 point) Calculate $\lim_{x\to\infty}(f(e^x))^{\frac{1}{x}}$;
(1 point) Represent graphically using numerical methods $g: R \to R$, $\frac{1}{2x^2} \cdot f(x)$; $g(x) = \frac{1}{x^2+1}$ [12].
Solutions.

$2A_2 = 2 \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \Rightarrow det(2A_2) = 12$, (1 point)

$A_3 + xI_3 = \begin{pmatrix} 2+x & 1 & 1 \\ 1 & 2+x & 1 \\ 1 & 1 & 2+x \end{pmatrix} \Rightarrow det(A + xI_3) = (x+4)(x+1)^2. Det(A + xI_3) = 0 \Rightarrow x \in \{-4, -1\}$, (1 point).

Det.A_4=5 different from 0 so A_4 is reversed. By direct calculation AB=BA=I_4, so B=A^{-1}, (1 point).

Whether $B' = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$ system X·B=C It has solutions $x_1=x_2=x_3=x_4=2$ (1 point).

$\lim_{x\to\infty} f(x) = \infty, \lim_{x\to\infty} \frac{f(x)}{x} = 2 = m, \lim_{x\to\infty}(f(x) - 2x) = 0 = n$, then $y = 2x$ oblique asymptote toward∞, (1 point).

$f'(x) = \frac{2x^4+6x^2}{(x^2+1)^2} \geq 0$, then f is increasing so injectivă . $\lim_{x\to-\infty} f(x) = -\infty, \lim_{x\to\infty} f(x) = \infty$ and f continues that f is surjective. Since f is bijective it follows that f is reversed. (1 point).

$\lim_{x\to\infty}(f(e^x))^{\frac{1}{x}} = \lim_{x\to\infty} 2^{\frac{1}{x}} \cdot e^3 \cdot (e^{2x} + 1)^{\frac{1}{x}} = e$, (1 point)
Graph of the function g (x) (1 point)

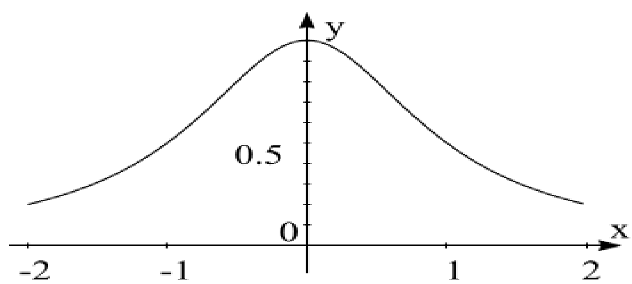


Fig. 1. The graphical representation of the function: $\mathbf{g(x) = \frac{1}{1+x^2}, x \in [-2, 2]}$

The analysis of the analytical and synthetic tables, the histogram, the frequency polygon and the circular diagram has revealed the following results of the experimental group in the final evaluation: of the 60 evaluated students, 64% of them obtained the mark VW (very well), 29% of the participants obtained the mark W (well), and 7% of the participants obtained the mark S (sufficient).

The analysis of the analytical and synthetic tables, the histogram, the frequency polygon and the circular diagram has revealed the following results of the control group in the final evaluation: of the 60 evaluated students, 44% of them obtained the mark VW (very well), 37% of the participants obtained the mark W (well), and 19% of the participants obtained the mark S (sufficient).

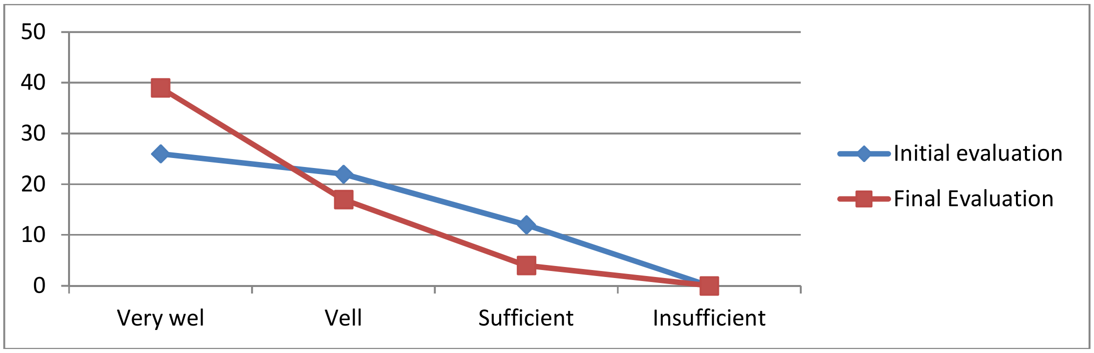
4.4 Comparative Analysis of the Data Obtained in the Initial and Final Evaluation

To highlight the progress achieved in improving relations as a result of the conducted experiment and the applied methodology, there was performed a comparative analysis of the two series for the initial and final evaluations.

Comparative analysis for the experimental group

Synthetic Table 1. Comparative analysis of the initial and final evaluation results

Marks	Initial evaluation	Final evaluation
Very well	26	39
Well	22	17
Sufficient	12	4
Insufficient	0	0



Frequency polygon 1. Comparative analysis of the initial and final evaluation results

The comparison between the predictive and the final test has revealed the fact that the systematic application of active methods and differentiated training during lessons throughout the entire school year generated both qualitative and quantitative progress.

The comparative analysis of the table and frequency polygon no. 1 reveals the progress achieved by the experimental group, by the end of the experiment. Thus, whereas in the initial stage there were 26 children with the mark of VW, representing 43%, at the stage of final evaluation their number grew to 39, representing 64% of the tested students. There was also an increase in the number of students who obtained the mark of W, as well as a decrease in the number of those who obtained the mark of S. The latter need further individual work to eliminate the gaps from their knowledge.

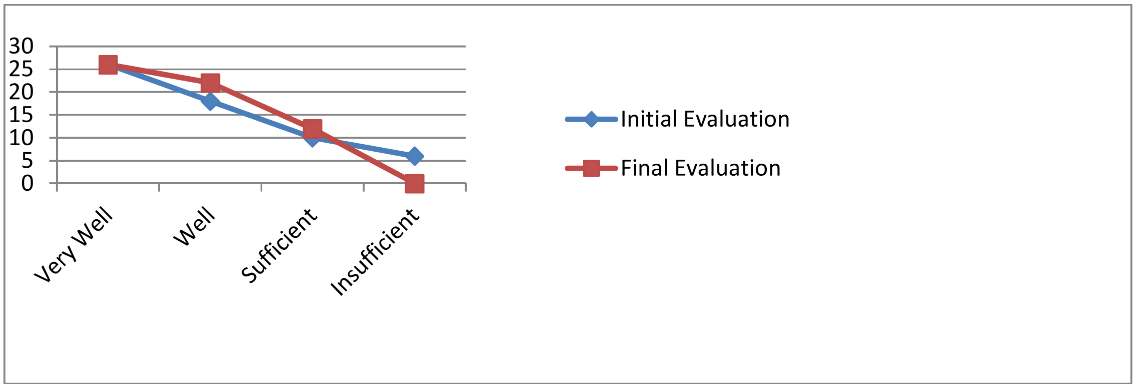
The results obtained in the final evaluation test demonstrate a clear difference from the scores obtained at the stage of the initial evaluation. This reveals the fact that the formative stage was efficient, the results obtained demonstrating an improvement in results.

The comparative analysis of Table no. 2 and frequency polygon no. 2 reveals, for the control group, the same number of students who obtained the mark of VW (44% in the initial stage, 44% in the final stage), a decrease in the number of the students who obtained the mark of W (44% in the initial stage, 37% in the final stage) and an increase in the number of the students who obtained the mark of S (12% in the initial stage, 19% in the final stage), there being no child with the mark of I.

Comparative analysis for the control group

Synthetic Table 2. Comparative analysis of the initial and final evaluation results

Marks	Initial evaluation	Final evaluation
Very well	26	26
Well	18	22
Sufficient	10	12
Insufficient	6	0



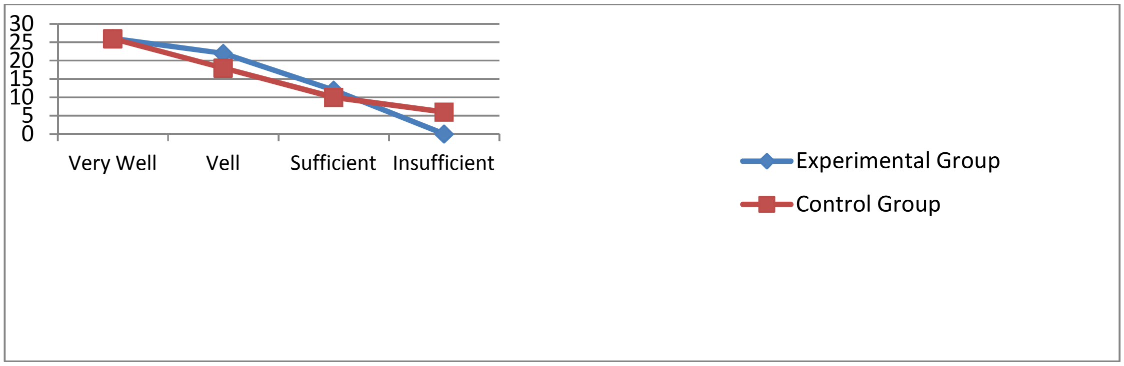
Frequency polygon 2. Comparative analysis of the initial and final evaluation results

Comparative analysis of the two groups, initial evaluation

The students from the two classes obtained, in the initial evaluation test, the following results:

Synthetic Table 3. Comparative analysis for the initial evaluation experimental group and control group

Marks	Experimental group	Control group
Very well	26	26
Well	22	18
Sufficient	12	10
Insufficient	0	6



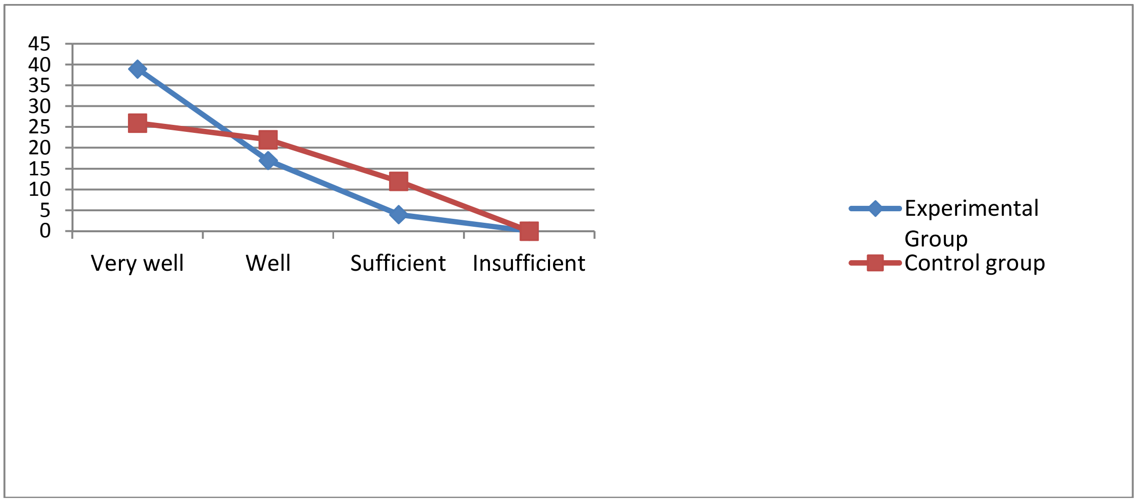
Frequency polygon 3. Graphical representation of the comparative analysis for the initial evaluation results experimental group and control group

The results show that the two classes of students have close results.

Final evaluation

Synthetic Table 4. Comparative analysis of the final evaluation results experimental group and control group

Marks	Experimental group	Control group
Very well	39	26
Well	17	22
Sufficient	4	12
Insufficient	0	0



Frequency polygon 4. Comparative analysis for the final evaluation experimental group and control group

The comparative analysis of the histogram and frequency polygon no. 4 reveals the progress achieved by the experimental group by the end of the experiment.

5. CONCLUSIONS

At the theoretical level, there were created the premises for conceptualizing the teaching strategies for solving mathematical problems. The analysis of contemporary theories and approaches supports the formulation of a series of conclusions:

- They hold a privileged position among the factors responsible for the students' school success;
- Using active and cooperation methods during lessons of Mathematics, and not only, has a favorable impact upon students as they contribute to building communication and team-work skills;
- The active-participative methods constitute a challenge, a curiosity for students and teachers alike.

Following the application of active and cooperation methods, we have found that their use for a specific purpose, at the right moment, may generate satisfying results, such as:

- The students overcame their communication barriers;
- They built skills in numerical solving of linear systems;
- They manifested appropriate behaviour in relation to their group mates;
- The students collaborated better and became more tolerant;
- The combination of forms of organization (frontal, group, individual) built wider possibilities for The students' multiple and diverse mobilization;
- The students learnt that they need one another to achieve group tasks.

The results obtained by the students confirm the research hypothesis. It was verified that the use of active-participative methods in the activity of solving problems of arithmetic contributes to optimizing learning and enhancing its efficiency, stimulating the students' intellectual and creative potential, obtaining performances according to age and individual particularities.

Regarding the initial evaluation, the mean of the class results were 6.42, an average that corresponds to below standard performances. In the final evaluation, the class average was 7.56, showing a relevant increase of **1,14** points. It is worth mentioning that a first progress was observed as of the stage of applying the experimental factor, the mean of the marks obtained by students in the formative evaluation being 6,85.

In conclusion, we may say that in order to achieve high-quality education and obtain the best results, we should combine classical and modern teaching, learning and evaluation methods, different forms of organization and a variety of teaching tools and materials.

COMPETING INTERESTS

Author has declared that no competing interests exist.

REFERENCES

1. Donev A. Numerical methods I solving nonlinear equations. Courant Institute of Mathematical Sciences, New York University, New York), Lecture VI. 2010;1–31.
2. Dumitreasa GH, Postolică V, Țifui V. Lessons of mathematical analysis. Matrix Rom Press, Bucharest; 2001.
3. Frumuşanu G. Numerical methods. University, Lower Danube; 2008. Galaţi.
4. Puiu G, Gîrţu M. General mathematics, Volume I, Technical Publishing, Scientific and Teaching, Iasi; 2004.
5. Heath MT. Scientific computing: An introductory survey, Chapter 5 – Nonlinear Equations, Department of Computer Science, University of Illinois at Urbana-Champaign; 2002.
6. Lupu C. The efficiency of computers with maple software in the teaching and learning of plane geometry. Social Media in Academia: Research and Teaching. 2015;215-221. Available:http://www.medimond.com/proceedings/moreinfo/20140918_index.pdf
7. Lupu C. The importance of new technologies in learning mathematics. Social Media in Academia - Research and Teaching. 2013;197-200. Available:http://www.medimond.com/proceedings/moreinfo/20130607_index.pdf
8. Lupu C. The contribution of the new technologies to learning mathematics. Procedia - Social and Behavioral Sciences. 2014;128:240-245. Available:http://www.sciencedirect.com/science/article/pii/S1877042814022411
9. Available:http://tutorial.math.lamar.edu/classes/alg/nonlinearsystems.aspx/11.04.2016
10. Lupu C. The role of the computer in learning mathematics through numerical methods. Science Journal of Education. 2016;4(2):32-38. Available:http://article.sciencepublishinggroup.com/html/10.11648.j.sjedu.20160402.13.html
11. Postolica V, Lupu C. Euclidean geometry and computers. International Journal of Applied Science and Mathematics. 2015;2(1):1-6. Available:http://www.ijasm.org/administrator/components/com_jresearch/files/publications/IJASM-27_final.pdf
12. Mariș S, Brăescu L. Numerical methods. Issues seminar and laboratory work, West University of Timisoara, Faculty of Mathematics and Informatics; 2007.

Peer-review history:
The peer review history for this paper can be accessed here:
http://sciencedomain.org/review-history/14574