A Collocation Method for Solving System of Volterra-differential-difference Equations with Terms of Chebyshev Polynomials

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this study, we present a numerical algorithm for solving systems of Volterra-differential-difference equations with variable coefficients by collocation method. This algorithm based on polynomial approximation, using the first kind Chebyshev polynomial basis with collocation method. This method transforms the system of Volterra-differential-difference equations and the given conditions into matrix equation which corresponds to a system of linear algebraic equation. In addition, convergence analysis of the method is presented. Some cases of the mentioned equations are solved as examples to illustrate the reliability of the method. The results reveal that the method is very effective and accuracy.

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1. INTRODUCTION

Systems of linear integral equations and their solutions are great importance in science and engineering [1–3]. Most physical and biological problems, such as biological applications in population dynamics and genetics can be modeled by the differential equation, an integral equation or an integro-differential equation or a system of these equations [4,5]. Moreover, the competition between tumor cells and the immune system, electromagnetic theory lead to the problem of solving integro-differential equation systems [6].

The systems of integral and integro-differential equations are usually difficult to solve analytically; so a numerical method is needed. Nowadays, systems of the integral and integro-differential equations have been solved using Euler matrix method [7], Chebyshev polynomial solution [8,9],

Bernstein operational matrix method [10], Tau method [11], homotopy perturbation method [12-14], Euler matrix method [15], Taylor collocation method [16] and others [17-22].

Chebyshev polynomials are encountered in several areas of numerical analysis and they hold particular importance in various subjects such as orthogonal polynomials, polynomial approximation, numerical integration and spectral methods [23-25]. Moreover, It is interesting to note that they also play an important part in the representation theory of algebras and polynomial factorization [26-28].

The aim of this paper is to develop the Chebyshev collocation method with matrix relations for a linear system of Volterra-differential-difference equations: for $j = 0,1,\ldots m$

$$\sum_{p=0}^{l}\sum_{k=0}^{m} H_{pk}^{j}(t) y_{k}^{(p)}(t) + \sum_{s=0}^{n}\sum_{k=0}^{m} P_{sk}^{j}(t) y_{k}^{(s)}(a_s t - b_s) = \sum_{v=0}^{m}\sum_{k=0}^{m}\int_{0}^{x} F_{vk}^{j}(t,x) y_{k}(x)dt + f_{j}(t) \qquad (1)$$

with conditions

$$\sum_{p=0}^{l-1}\sum_{q=0}^{m} c_{pq}^{j} y_{q}^{(p)}(a_{p}^{j}) = \lambda_{j}, \ \ j = 0,1,\ldots,n \qquad (2)$$

where a_s , b_s , c_{pq}^{j} and λ_i are constant and $-1 \le a_{pq}^{j} \le 1$. $f_j(t)$, $H_{pk}^{j}(t)$, $P_{sk}^{j}(t)$ and $F_{vk}^{j}(t,x)$ are analytic functions and $y_k(x)$ unknown functions. For numerical solution of Eq.(1) with initial conditions Eq.(2), we construct to the Chebyshev series solutions that is;

$$y_{j}^{N}(t) = \sum_{r=0}^{N} a_{r}^{j} T_{r}(t), \ T_{r}(t) = \cos(r\theta), \ t = \cos\theta, \ j = 0,1,\ldots m \qquad (3)$$

where $T_r(t)$ denotes the Chebyshev polynomials of the first kind, a_r^{j} $(0 \le r \le N)$ are unknown Chebyshev coefficients, and N is chosen any positive integer.

Collocation method with matrix relations have been used for solving differential-difference, pantograph equations, Fredholm-Volterra differential equations, singular equations, fractional differential and pantograph equation by some authours [29-43].

The Chebyshev polynomials $T_r(t)$ of the first kind are the polynomials in t of degree r , defined by relation [23-25]

$T_r(t) = \cos n\theta$, when $t = \cos\theta$

If the range of the variable t is the interval $[-1,1]$, the range the corresponding variables θ can be taken $[0,\pi]$. These polynomials have the following properties [23-25]:

i) $T_{r+1}(t)$ has exactly $r+1$ real zeroes on the interval $[-1,1]$ (The Chebyshev-Gaus grid points). The i-th zero t_i of $T_{r+1}(t)$ is located at

$$t_i = \cos\frac{(2(r-i)+1)\pi}{2(r+1)}, \ i = 0,1,\ldots,N \quad (4)$$

ii) $T_r(t)$ is ortogonal on $[-1,1]$ with respect to the weight function $w(t) = (1-t^2)^{-\frac{1}{2}}$.

iii) It is well known that the relation between the powers t^n and the Chebyshev polynomials $T_r(t)$ is

$$t^{2r} = 2^{-2r+1}\sum_{s=0}^{r}\binom{2r}{r-s}T_{2s}(t) \quad (5)$$

$$t^{2r+1} = 2^{-2r}\sum_{s=0}^{r}\binom{2r+1}{r-s}T_{2s+1}(t) \quad (6)$$

2. FUNDAMENTAL MATRIX RELATIONS

To solve Eq.(1), we construct the following matrix realation. Using the Eq.(3), we have the matrix relations of solutions

$$y_j^N(t) = \mathbf{T}(t)\mathbf{A}^j, \ j = 0,1,\ldots m \quad (7)$$

Where

$$\mathbf{T}(t) = [T_0(t) \ T_1(t) \ldots T_N(t)]$$
$$\mathbf{A}^j = [a_0^j \ a_1^j \ldots a_N^j]^T$$

By using the expression (5-6) and taking $r = 0,1,\ldots,N$ we find the corresponding matrix relation as follows

$$(\mathbf{Y}(t))^T = \mathbf{D}(\mathbf{T}(t))^T \qquad \mathbf{Y}(t) = \mathbf{T}(t)\mathbf{D}^T \ (8)$$

where

$$\mathbf{Y}(t) = [1 \ t \ldots t^N]$$

and for odd N,

$$\mathbf{D} = \begin{bmatrix} \binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \binom{N}{(N-1)/2} & 0 & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}$$

for even N,

$$\mathbf{D} = \begin{bmatrix} \binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ \binom{N}{N/2}2^{1-N} & 0 & \binom{N}{(N-2)/2}2^{1-N} & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}$$

Then, by taking into account (8) we obtain

$$\mathbf{T}(t) = \mathbf{Y}(t)(\mathbf{D}^{-1})^T$$

and

$$(\mathbf{T}(t))^{(p)} = \mathbf{Y}^{(p)}(t)(\mathbf{D}^{-1})^T \ , \quad p = 0,1,\ldots,l \ (9)$$

To obtain the matrix $\mathbf{Y}^{(k)}(t)$ in terms of the matrix $\mathbf{Y}(t)$, we can use the following relation:

$$\mathbf{Y}^{(p)}(t) = \mathbf{Y}(t)(\mathbf{B}^T)^p, \ p = 0,1,\ldots,l \quad (10)$$

where $(\mathbf{B}^T)^0 = I_{(n+1)\times(n+1)}$ and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}$$

Consequently, by substituting the matrix forms (10) and (11) into (9) and its derivatives, we get the approximate solution and its first-derivative of the matrix relations

$$y_j^N(t) = \mathbf{Y}(t)(\mathbf{D}^T)^{-1}\mathbf{A}^j \text{ and}$$

$$\left(y_j^N(t)\right)^{(p)} = \mathbf{Y}(t)(\mathbf{B}^T)^p(\mathbf{D}^T)^{-1}\mathbf{A}^j \qquad (11)$$

Moreover, Using Eq.(7) and Eq.(9), we obtain the following matrix representation of $y_j^N(a_s t - b_s)$:

$$\left(y_j^N\right)^{(s)}(a_s t - b_s) = \mathbf{T}^{(s)}(a_s t - b_s)\mathbf{A}^j = \mathbf{Y}^{(s)}(a_s t - b_s)(\mathbf{D}^T)^{-1}\mathbf{A}^j, \quad j = 0,1,\dots m \qquad (12)$$

and using Binomial expansion, we obtain the following matrix relation between the matrices $\mathbf{Y}(t - v_s)$ and $\mathbf{Y}(t)$:

$$\mathbf{Y}^{(s)}(a_s t - b_s) = \mathbf{Y}(t)\left(\mathbf{B}^T\right)^s \mathbf{B}_s \qquad (13)$$

Where

$$\mathbf{B}_s = \begin{bmatrix} \binom{0}{0}a_s^N(-b_s)^0 & \binom{1}{0}a_s^{N-1}(-b_s)^1 & \binom{2}{0}a_s^{N-2}(-b_s)^2 & \cdots & \binom{N}{0}a_s^0(-b_s)^N \\ 0 & \binom{1}{1}a_s^{N-1}(-b_s)^0 & \binom{2}{1}a_s^{N-2}(-b_s)^1 & \cdots & \binom{N}{1}a_s^0(-b_s)^{N-1} \\ 0 & 0 & \binom{2}{2}a_s^{N-2}(-b_s)^0 & \cdots & \binom{N}{2}a_s^0(-b_s)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}a_s^0(-b_s)^0 \end{bmatrix}$$

So, using Eq.(12) and Eq.(13), we have the matrix relations of $y_j^N(a_s t - b_s)$:

$$\left(y_j^N(a_s t - b_s)\right)^{(s)} = \mathbf{Y}(t)\left(\mathbf{B}^T\right)^s \mathbf{B}_s(\mathbf{D}^T)^{-1}\mathbf{A}^j, \quad j = 0,1,\dots m \qquad (14)$$

2.1 Matrix Representation of Volterra Integral Part

Let assume that $F_{vk}^j(t, x)$ can be expanded to univariate Chebyshev series with respect to *t* as follows:

$$F_{vk}^j(t, x) = \sum_{r=0}^{N} f_{vkr}^j(t)T_r(x). \qquad (15)$$

Then the matrix representations of the kernel function $F_{vk}^j(t, x)$ become

$$F_{vk}^j(t, x) = \mathbf{F}_{vk}^j(t)\mathbf{T}^T(x) \qquad (16)$$

where

$$\mathbf{F}_{vk}^j(t) = [f_{vk0}^j(t) \quad f_{vk1}^j(t) \quad f_{vk2}^j(t) \quad \cdots \quad f_{vkN}^j(t)].$$

Substitutig the relations (6) and (16) in integral part, we obtained

$$\sum_{v=0}^{h}\sum_{k=0}^{m}\int_0^t F_{vk}^j(t,x)y_k(x)dx = \sum_{v=0}^{h}\sum_{k=0}^{m}\int_0^t F_{vk}^j(t)\mathbf{T}^T(x)\mathbf{T}(x)\mathbf{A}^k dx$$

$$= \sum_{v=0}^{h}\sum_{k=0}^{m}\int_0^t F_{vk}^j(t)\mathbf{D}^{-1}\mathbf{Y}^T(x)\mathbf{Y}(x)(\mathbf{D}^T)^{-1}\mathbf{A}^k dx = \sum_{v=0}^{h}\sum_{k=0}^{m}F_{vk}^j(t)\mathbf{D}^{-1}\left(\int_0^t \mathbf{Y}^T(x)\mathbf{Y}(x)dx\right)(\mathbf{D}^T)^{-1}\mathbf{A}^k$$

$$= \sum_{v=0}^{h}\sum_{k=0}^{m}F_{vk}^j(t)\mathbf{D}^{-1}\mathbf{Q}(t)(\mathbf{D}^T)^{-1}\mathbf{A}^k \qquad (17)$$

where

$$\mathbf{Q}(t) = \int_0^t \mathbf{Y}^T(x)\mathbf{Y}(x)dt$$

and

$$\mathbf{Q}(t) = [q_{ij}] = \frac{t^{i+j+1}}{i+j+1}, \quad i,j = 0,1,\ldots,N.$$

3. METHOD OF SOLUTION

We are now ready to construct the fundamental matrix equation corresponding to Eq.(1). For this purpose, we substitute the matrix relations Eqs.(11),(14),(17) into Eq.(1) and obtain the matrix equation:

$$\sum_{p=0}^{l}\sum_{k=0}^{m}H_{pk}^j(t)\mathbf{Y}(t)(\mathbf{B}^T)^p(\mathbf{D}^T)^{-1}\mathbf{A}^k + \sum_{s=0}^{n}\sum_{k=0}^{m}P_{sk}^j(t)\mathbf{Y}(t)(\mathbf{B}^T)^s\mathbf{B}_s(\mathbf{D}^T)^{-1}\mathbf{A}^k$$

$$= f_j(t) + \sum_{v=0}^{h}\sum_{k=0}^{m}F_{vk}^j(t)\mathbf{D}^{-1}\mathbf{Q}(t)(\mathbf{D}^T)^{-1}\mathbf{A}^k \qquad (18)$$

then, it can be written as:

$$\left(\sum_{p=0}^{n}\mathbf{H}_p(t)\overline{\mathbf{Y}(t)}\overline{(\mathbf{B}^T)^p}\mathbf{D} + \sum_{s=0}^{n}\mathbf{P}_s(t)\overline{\mathbf{Y}(t)}\overline{(\mathbf{B}^T)^s}\overline{\mathbf{B}_s}\mathbf{D} - \sum_{v=0}^{h}\mathbf{F}_v(t)\overline{\mathbf{D}^{-1}}\mathbf{Q}(t)\mathbf{D}\right)\mathbf{A} = \mathbf{F} \qquad (19)$$

where

$$\overline{\mathbf{Y}(t)} = \begin{bmatrix} \mathbf{Y}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{Y}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Y}(t) \end{bmatrix} \quad \overline{(\mathbf{B}^T)^p} = \begin{bmatrix} (\mathbf{B}^T)^p & 0 & \cdots & 0 \\ 0 & (\mathbf{B}^T)^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mathbf{B}^T)^p \end{bmatrix}$$

$$\overline{\left(\mathbf{B}^T\right)^s} = \begin{bmatrix} \left(\mathbf{B}^T\right)^s & 0 & \cdots & 0 \\ 0 & \left(\mathbf{B}^T\right)^s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \left(\mathbf{B}^T\right)^s \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} (\mathbf{D}^T)^{-1} & 0 & \cdots & 0 \\ 0 & (\mathbf{D}^T)^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\mathbf{D}^T)^{-1} \end{bmatrix}$$

$$\mathbf{P}_s(t) = \begin{bmatrix} P_{s0}^0(t) & P_{s1}^0(t) & \cdots & P_{sn}^0(t) \\ P_{s0}^1(t) & P_{s1}^1(t) & \cdots & P_{sn}^1(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{s0}^m(t) & P_{s1}^m(t) & \cdots & P_{sn}^m(t) \end{bmatrix} \quad \mathbf{H}_p(t) = \begin{bmatrix} H_{p0}^0(t) & H_{s1}^0(t) & \cdots & H_{pl}^0(t) \\ H_{p0}^1(t) & H_{s1}^1(t) & \cdots & H_{pl}^1(t) \\ \vdots & \vdots & \ddots & \vdots \\ H_{p0}^m(t) & H_{p1}^m(t) & \cdots & H_{pl}^m(t) \end{bmatrix}$$

$$\overline{\mathbf{B}_s} = \begin{bmatrix} \mathbf{B}_s & 0 & \cdots & 0 \\ 0 & \mathbf{B}_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}_s \end{bmatrix} \quad \overline{\mathbf{D}^{-1}} = \begin{bmatrix} \mathbf{D}^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{D}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^{-1} \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} f_0(t) \\ f_1(t) \\ \vdots \\ f_m(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}^0 \\ \mathbf{A}^1 \\ \vdots \\ \mathbf{A}^m \end{bmatrix}$$

$$\overline{\mathbf{F}}_v(t) = \begin{bmatrix} \mathbf{F}_{v0}^0(t) & \mathbf{F}_{v1}^0(t) & \cdots & \mathbf{F}_{vm}^0(t) \\ \mathbf{F}_{v0}^1(t) & \mathbf{F}_{v1}^1(t) & \cdots & \mathbf{F}_{vm}^1(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}_{v0}^m(t) & \mathbf{F}_{v1}^m(t) & \cdots & \mathbf{F}_{vm}^m(t) \end{bmatrix}$$

$$\mathbf{Q}(t) = \begin{bmatrix} t & t^2/2 & \cdots & t^{N+1}/N+1 \\ t^2/2 & t^3/3 & \cdots & t^{N+2}/N+2 \\ \vdots & \vdots & \ddots & \vdots \\ t^{N+1}/N+1 & t^{N+2}/N+2 & \cdots & t^{2N+1}/2N+1 \end{bmatrix}$$

When the points of Chebyshev-Gauss grid are substituting in Eq.(16), the fundamental matrix equation of the system of Fredholm-differential-difference equations Eq.(1) is

$$\left(\sum_{p=0}^n \overline{\mathbf{H}}_p \ \overline{\mathbf{Y}} \ \overline{\left(\mathbf{B}^T\right)^p} \ \mathbf{D} + \sum_{s=0}^n \overline{\mathbf{P}}_s \ \overline{\mathbf{Y}} \ \overline{\left(\mathbf{B}^T\right)^s} \ \overline{\mathbf{B}}_s \ \mathbf{D} - \sum_{v=0}^h \overset{*\_*}{\mathbf{F}}_v \overset{*\_*\_*}{\mathbf{D}^{-1}} \overset{*\_*\_*}{\mathbf{Q}} \overset{*\_*}{\mathbf{D}} \right) \mathbf{A} = \mathbf{F} \tag{20}$$

where

$$\overset{*\_*}{\mathbf{F}}_v = \begin{bmatrix} \overline{\mathbf{F}}_v(t_0) \\ \overline{\mathbf{F}}_v(t_1) \\ \vdots \\ \overline{\mathbf{F}}_v(t_N) \end{bmatrix} \quad \overset{*\_*\_*}{\mathbf{D}^{-1}} = \begin{bmatrix} \overline{\mathbf{D}^{-1}} & 0 & \cdots & 0 \\ 0 & \overline{\mathbf{D}^{-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\mathbf{D}^{-1}} \end{bmatrix} \quad \overset{*\_*}{\mathbf{D}} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D} \\ \vdots \\ \mathbf{D} \end{bmatrix}$$

$$\overset{*\_*}{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}(t_0) & 0 & \cdots & 0 \\ 0 & \mathbf{Q}(t_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Q}(t_N) \end{bmatrix}$$

where

$$\overline{\mathbf{Y}} = \begin{bmatrix} \overline{\mathbf{Y}(t_0)} & 0 & \cdots & 0 \\ 0 & \overline{\mathbf{Y}(t_1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\mathbf{Y}(t_N)} \end{bmatrix} \quad \overline{\mathbf{P}_s} = \begin{bmatrix} \mathbf{P}_{s0} & 0 & \cdots & 0 \\ 0 & \mathbf{P}_{s1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_{sN} \end{bmatrix}$$

$$\overline{\mathbf{H}_p} = \begin{bmatrix} \mathbf{H}_{p0} & 0 & \cdots & 0 \\ 0 & \mathbf{H}_{p1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_{pN} \end{bmatrix} \quad \overline{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_N \end{bmatrix}$$

and

$$\overline{\mathbf{Y}(t_i)} = \begin{bmatrix} \mathbf{Y}(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{Y}(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Y}(t_i) \end{bmatrix} \quad \mathbf{H}_{pi} = \begin{bmatrix} \mathbf{H}_p(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{H}_p(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{H}_p(t_i) \end{bmatrix}$$

$$\mathbf{P}_{si} = \begin{bmatrix} \mathbf{P}_s(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_s(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_s(t_i) \end{bmatrix} \quad \mathbf{F}_i = \begin{bmatrix} f_0(t_i) \\ f_1(t_i) \\ \vdots \\ f_m(t_i) \end{bmatrix}$$

Hence, the matrix equation (17) correponding to Eq.(1) can be written in the form

$$\mathbf{WA} = \overline{\mathbf{F}} \text{ or } \left[\mathbf{W}; \overline{\mathbf{F}} \right] \tag{21}$$

where

$$\mathbf{W} = \sum_{p=0}^{n} \overline{\mathbf{H}_p} \ \overline{\mathbf{Y}} \ \overline{\left(\mathbf{B}^T\right)^p} \mathbf{D} + \sum_{s=0}^{n} \overline{\mathbf{P}_s} \ \overline{\mathbf{Y}} \ \overline{\left(\mathbf{B}^T\right)^s} \ \overline{\mathbf{B}_s} \ \mathbf{D} - \sum_{v=0}^{h} \overline{\mathbf{F}_v} \ \overline{\mathbf{D}^{-1}} \ \overline{\mathbf{Q}} \ \overline{\mathbf{D}}$$

Here, Eq.(21) corresponding to a system of $m(N+1)$ linear algebraic equations with $m(N+1)$ unknown Chebyshev coefficients.

We can obtain the corresponding matrix forms for conditions (2), by means of the relation (11), as

$$\sum_{p=0}^{l-1}\sum_{q=0}^{m} c_{pq}^j \ \overline{\mathbf{Y}(a_p^j)}\overline{\left(\mathbf{B}^T\right)^p} \mathbf{DA} = \left[\lambda_j\right],$$
$$-1 \le a_p^j \le 1 \ , \ j = 0,1,\ldots,m \tag{22}$$

On the other hand, the matrix form for conditions can be written as

$$\mathbf{UA} = \mathbf{G} \tag{23}$$

where for $a = 0,1,\ldots,m$, $b = 0,1,\ldots,m(N+1)$

$$\mathbf{U} = [u_{ab}] = \begin{bmatrix} \mathbf{U}_0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{U}_1 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{U}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{U}_m \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix}$$

and

$$\mathbf{U}_j = \sum_{p=0}^{l-1} \sum_{q=0}^{m} c_{pq}^j \, \overline{\mathbf{Y}(a_p^j)} \left(\mathbf{B}^T\right)^p \mathbf{D} \quad, \\ j = 0,1,\dots,m$$

We can obtain the approximate solutions of Eq.(1) with the conditions Eq.(2) by terms of Chebyshev polynomials. By replacing the conditions matrices (21) by the last l rows of the matrix (23) we obtain the new augmented matrix

$$\overline{\overline{\mathbf{W}}} \, \mathbf{A} = \overset{\bullet}{\mathbf{F}} \tag{24}$$

where

$$\overline{\overline{\mathbf{W}}} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0\times m(N+1)} \\ w_{10} & w_{11} & \cdots & w_{1\times m(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m(N-1)\times 0} & w_{m(N-1\times)1} & \cdots & w_{m(N-1)\times m(N+1)} \\ u_{00} & u_{01} & \cdots & u_{0\times m(N+1)} \\ \vdots & \vdots & \cdots & \vdots \\ u_{m0} & u_{m1} & \cdots & u_{m\times m(N+1)} \end{bmatrix}$$

and

$$\overset{\bullet}{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_0 \\ \vdots \\ \mathbf{F}_{N-1} \\ \mathbf{G} \end{bmatrix}$$

So, we obtained to a system of $m(N+1) \times m(N+1)$ linear algebraic equations with $m(N+1)$ unknown Chebyshev coefficients. If $rank\,\overline{\overline{\mathbf{W}}} = rank[\overline{\overline{\mathbf{W}}};\overset{\bullet}{\mathbf{F}}] = m(N+1)$, the we can be write $\mathbf{A} = (\overline{\overline{\mathbf{W}}})^{-1}\overset{\bullet}{\mathbf{F}}$. Thus, the matrix $\mathbf{A}$

(thereby the coefficients matrix $\mathbf{A}^j$, $j = 0,1,\dots,m$) is uniquely determined. Also the Eq.(1) with conditions (2) has a unique solution. On the other hand, when $\left|\overline{\overline{\mathbf{W}}}\right| = 0$, if $rank\,\overline{\overline{\mathbf{W}}} = rank[\overline{\overline{\mathbf{W}}};\overset{\bullet}{\mathbf{F}}] < m(N+1)$, then we may find a particular solution. Otherwise if $rank\,\overline{\overline{\mathbf{W}}} \neq rank[\overline{\overline{\mathbf{W}}};\overset{\bullet}{\mathbf{F}}] < m(N+1)$, then it is not a solution.

3.1 Convergence Analysis

In this section, we present convergence analysis of the mention method. We assume that $y(t)$ is a sufficiently smooth function on $[-1,1]$ and $I_N(t)$ is the interpolating polynomial to y at t_i, where t_i, $i = 0,1,\dots,n$ are the Cbeyshev-Gauss grid points, then we have

$$y(t) - I_N(t) = \frac{y^{(N+1)}(\lambda)}{(n+1)!} \prod_{i=0}^{N} (t - t_i), \lambda \in [0,1]$$

Therefore, we have [23,25]

$$\left| y(t) - I_N(t) \right| \leq \frac{1}{2^{N+1}(N+1)!} \left\| y^{(N+1)}(t) \right\|_\infty \tag{25}$$

Theorem 1. Suppose that the known functions in Eq.(1) are real $(N+1)$-times continuosly differential functions on the $[-1,1]$ and

$$\overline{y_N^j}(t) = \sum_{r=0}^{N} \overline{a_r^j}\, T_r(t), \quad j = 0,1,2,\dots,n$$

are the shifted Chebshev polynomials expansion of the exact solution.

Let

$$y_N^j(t) = \sum_{r=0}^{N} a_r^j T_r(t)$$

be the approximate solution obtained by proposed method, then

$$\left\| y^j(t) - y_N^j(t) \right\|_2 \le \frac{1}{2^{N+1}} \left\| (y^j(t))^{(N+1)} \right\|_\infty + \sqrt{\frac{3\pi}{8}} \left\| \mathbf{A}^j - \overline{\mathbf{A}}^j \right\|_2$$

where

$$\mathbf{A}^j = \begin{bmatrix} a_0^j & a_1^j & \cdots & a_N^j \end{bmatrix} \text{ ve } \overline{\mathbf{A}}^j = \begin{bmatrix} \overline{a_0^j} & \overline{a_1^j} & \cdots & \overline{a_N^j} \end{bmatrix}$$

Proof: Let $y_N(t)$ is real-valued polynomials of degree $\le N$ and $y_N(t)$ is the best aprroximation of $y(t)$. We can write

$$\left\| y^j(t) - y_N^j(t) \right\|_2 \le \left\| y^j(t) - \overline{y_N^j}(t) \right\|_2 + \left\| \overline{y_N^j}(t) - y_N^j(t) \right\|_2$$

Using (25) we obtain

$$\left\| y^j(t) - \overline{y_N^j}(t) \right\|_2 = \left(\int_0^1 \left| y(t) - \overline{y_N^j}(t) \right|^2 dt \right)^{1/2} \le \left(\int_0^1 \left[\frac{1}{2^{N+1}(N+1)!} \left\| (y^j(t))^{(N+1)} \right\|_\infty \right]^2 dt \right)^{1/2}$$

$$= \frac{1}{2^{2N+1}(N+1)!} \left(\left\| (y^j(t))^{(N+1)} \right\|_\infty \right)$$

and we have

$$\left\| y_N^j(t) - \overline{y_N^j}(t) \right\|_2 = \left(\int_0^1 \left[\sum_{r=0}^N (a_r^j - \overline{a_r^j}) T_r(t) \right]^2 dt \right)^{1/2} \le \left(\int_0^1 \left[\sum_{r=0}^N (a_r^j - \overline{a_r^j})^2 \right] \left[\sum_{r=0}^N |T_r(t)|^2 \right] dt \right)^{1/2}$$

$$= \left[\sum_{r=0}^N (a_r^j - \overline{a_r^j})^2 \right]^{1/2} \left(\sum_{r=0}^N \int_0^1 |T_r(t)|^2 dt \right)^{1/2} = \sqrt{\frac{3\pi}{8}} \left\| \mathbf{A}^j - \overline{\mathbf{A}}^j \right\|$$

We can easily check the accuracy of the method. Since the truncated Chebyshev series (3) is an approximate solutions of Eq.(1), when the function $y_j^N(t)$, $j = 0, 1, \ldots, m$ and its first derivatives are substituted in Eq. (1) the resulting equation must be satisfied approximately[25]; that is, for $t_i \in [-1,1]$, $i = 0, 1, 2, \ldots$

$$\left| \sum_{p=0}^l \sum_{k=0}^m H_{pk}^j(t_i) y_k^{N(p)}(t_i) + \sum_{s=0}^n \sum_{k=0}^m P_{sk}^j(t_i) y_k^{N(s)}(a_s t_i - b_s) - \sum_{v=0}^m \sum_{k=0}^m \int_0^x F_{vk}^j(t_i, x) y_k^N(x) dt - f_j(t_i) \right| \cong 0$$

On the other hand, the error can be estimated by the function [25]

$$E_j^N(t) = \sum_{p=0}^l \sum_{k=0}^m H_{pk}^j(t_i) y_k^{N(p)}(t_i) + \sum_{s=0}^n \sum_{k=0}^m P_{sk}^j(t_i) y_k^{N(s)}(a_s t_i - b_s) - \sum_{v=0}^m \sum_{k=0}^m \int_0^x F_{vk}^j(t_i, x) y_k^N(x) dt - f_j(t_i)$$

4. NUMERICAL EXAMPLES

In this section, we give some the numerical examples. All numerical scheme are calculated by using Maple 13. The absolute errors in Tables are the values of $N_e^j = \left| y_j(t) - y_j^N(t) \right|$, those at selected points. In Tables, $\max \left| y_j(t) - y_j^N(t) \right|$ is maximum absolute errors and $Cond(\overline{\mathbf{W}})$ is condition numbers of the matrix $\overline{\mathbf{W}}$.

Moreover, we compare the absolute errors and L^2-norm errors is defined by

$$E_N^L = \left(\int_0^1 \left(y(x) - y_N(x) \right)^2 dx \right)^{1/2}$$

where $y(x)$ and $y_N(x)$ denote the approximate solution obtained by the present method and the exact solution, respectively. In Tables, $N_e = \left| y(x) - y_N(x) \right|$ are absolute error for selected points.

Example 1. Consider the systems of Volterra-delay-differential equations:

$$y_0''(x) - e^{-x} y_1'(x) - y_0(x) + y_1(x) + y_0(x-0.5) - y_1(x-0.5) = \int_0^x y_0(t)dt + \int_0^x y_1(t)dt + f_0(x) \quad (26)$$

$$y_1''(x) + e^x y_0'(x) + y_0(x) - y_1(x) - y_0(x-0.5) + y_1(x-0.5) = \int_0^x y_0(t)dt - \int_0^x y_1(t)dt + f_1(x)$$

where

$$f_0(t) = e^{-t^2} + e^{-t} + e^{t-0.5} - e^{-t+0.5} - e + e^{-1}$$
$$f_1(t) = e^{t^2} + e^t - e^{t-0.5} + e^{-t+0.5} + 2 - e - e^{-1}$$

with conditions

$$y_0(0) = y_1(0) = y_0'(0) = 1, \ y_1'(0) = -1.$$

We assume that the problem (26) possess Chebyshev polynomial solutions in the form

$$y_j^5(t) = \sum_{r=0}^5 a_r^j T_r(t), \ j = 0,1$$

and

$$H_{00}^0(x) = -1 \ , \ H_{01}^0(x) = 1 \ , \ H_{11}^0(x) = -e^{-x} \ , \ H_{20}^0(x) = 1 \ , \ P_{00}^0(x) = 1 \ , \ P_{01}^0(x) = -1 \ ,$$
$$H_{00}^1(x) = 1, \ H_{01}^1(x) = 1, \ H_{10}^1(x) = e^x, \ H_{02}^1(x) = 1, \ P_{00}^1(x) = -1, \ P_{01}^1(x) = 1, \ F_{00}^0(x,t) = 1,$$
$$F_{01}^0(x,t) = 1, \ F_{00}^1(x,t) = 1, \ F_{01}^1(x,t) = -1.$$

From equations (29), we have

$$\mathbf{P}_0(t) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \quad \mathbf{P}_1(t) = \begin{bmatrix} 0 & -e^{-t} \\ e^t & 0 \end{bmatrix} \quad \mathbf{P}_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{H}_0(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F_{00}^0(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{01}^0(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{00}^1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F_{01}^1(t) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q} = \int_0^x \mathbf{Y}^T(x)\mathbf{Y}(x)dx$$

where

$$\mathbf{Q}(x) = [q_{ij}] = \frac{x^{i+j+1}}{i+j+1}, \quad i,j = 0,1,...,N.$$

and we can rewrite

$$\mathbf{F}_0(t) = \begin{bmatrix} \mathbf{F}_{00}^0(t) & \mathbf{F}_{01}^0(t) \\ \mathbf{F}_{00}^1(t) & \mathbf{F}_{01}^1(t) \end{bmatrix}$$

For $N = 5$, the Chebyshev-Gaus grid points are

$$t_0 = -\cos(\frac{\pi}{12}), \quad t_1 = -\frac{\sqrt{2}}{2}, \quad t_2 = -\cos(\frac{5\pi}{12}), \quad t_3 = \cos(\frac{\pi}{12}), \quad t_4 = \frac{\sqrt{2}}{2}, \quad t_5 = \cos(\frac{\pi}{12})$$

and the fundemental matrix form of the problem is defined by

$$\left(\overline{\mathbf{P}}_2 \, \overline{\mathbf{Y}} \, \overline{(\mathbf{B}^T)}^2 \, \mathbf{D} + \overline{\mathbf{P}}_1 \, \overline{\mathbf{Y}} \, \overline{(\mathbf{B}^T)} \mathbf{D} + \overline{\mathbf{P}_0} \, \overline{\mathbf{Y}} \, \mathbf{D} + \overline{\mathbf{H}}_0 \, \overline{\mathbf{Y}} \, \overline{\mathbf{B}}_0 \, \mathbf{D} - \overline{\mathbf{F}}_v \, \mathbf{D}^{-1} \, \overset{*_{-}* \, *_{-}*_{-}* \, *_{-}*}{\mathbf{D}} \right) \mathbf{A} = \overline{\mathbf{F}} \tag{27}$$

where

$$\overline{\mathbf{P}}_0 = \begin{bmatrix} \mathbf{P}_{00} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{P}_{02} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{P}_{03} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{P}_{04} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{05} \end{bmatrix} \quad \overline{\mathbf{P}}_1 = \begin{bmatrix} \mathbf{P}_{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{P}_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{P}_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{P}_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{15} \end{bmatrix}$$

$$\overline{\mathbf{P}}_2 = \begin{bmatrix} \mathbf{P}_{20} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{21} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{P}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{P}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{P}_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{25} \end{bmatrix} \quad \overset{*\_*}{\mathbf{F}_0} = \begin{bmatrix} \overline{\mathbf{F}_0}(x_0) \\ \overline{\mathbf{F}_0}(x_1) \\ \overline{\mathbf{F}_0}(x_2) \\ \overline{\mathbf{F}_0}(x_3) \\ \overline{\mathbf{F}_0}(x_5) \end{bmatrix}$$

$$\overset{*\_*\_*}{\mathbf{D}^{-1}} = \begin{bmatrix} \overline{\mathbf{D}^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \overline{\mathbf{D}^{-1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\mathbf{D}^{-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\mathbf{D}^{-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{\mathbf{D}^{-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{\mathbf{D}^{-1}} \end{bmatrix}$$

$$\overline{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \\ \mathbf{F}_4 \\ \mathbf{F}_5 \end{bmatrix} \quad \overset{*\_*}{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}(x_0) & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{Q}(x_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{Q}(x_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{Q}(x_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{Q}(x_4) & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{Q}(x_5) \end{bmatrix} \quad \overset{*\_*}{\mathbf{D}} = \begin{bmatrix} \mathbf{D} \\ \mathbf{D} \\ \mathbf{D} \\ \mathbf{D} \\ \mathbf{D} \\ \mathbf{D} \end{bmatrix}$$

$$\overline{\mathbf{B}^T} = \begin{bmatrix} \mathbf{B}^T & 0 \\ 0 & \mathbf{B}^T \end{bmatrix} \quad \overline{\mathbf{B}_0} = \begin{bmatrix} \mathbf{B}_0 & 0 \\ 0 & \mathbf{B}_0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} (\mathbf{D}^T)^{-1} & 0 \\ 0 & (\mathbf{D}^T)^{-1} \end{bmatrix} \quad \overline{\mathbf{D}^{-1}} = \begin{bmatrix} \mathbf{D}^{-1} & 0 \\ 0 & \mathbf{D}^T \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^0 \\ \mathbf{A}^1 \end{bmatrix}$$

and for $i = 0,1,2,...,5$

$$\mathbf{P}_{0i} = \mathbf{P}_0(t_i), \ \mathbf{P}_{1i} = \mathbf{P}_1(t_i), \ \mathbf{P}_{2i} = \mathbf{P}_2(t_i), \ \mathbf{F}_i = \begin{bmatrix} f_0(t_i) \\ f_1(t_i) \end{bmatrix}$$

$$(\mathbf{D}^T)^{-1} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -3 & 0 & 5 \\ 0 & 0 & 2 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & -20 \\ 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{bmatrix} \quad \mathbf{B}_0 = \begin{bmatrix} 1 & -0.5 & 0.25 & -0.125 & 0.0625 & -0.03125 \\ 0 & 1 & -1.0 & 0.75 & -0.5 & 0.3125 \\ 0 & 0 & 1 & -1.5 & 1.5 & -1.25 \\ 0 & 0 & 0 & 1 & -2.0 & 2.5 \\ 0 & 0 & 0 & 0 & 1 & -2.5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}(x) = \begin{bmatrix} x & x^2/2 & x^3/3 & x^4/4 & x^5/5 & x^6/6 \\ x^2/2 & x^3/3 & x^4/4 & x^5/5 & x^6/6 & x^7/7 \\ x^3/3 & x^4/4 & x^5/5 & x^6/6 & x^7/7 & x^8/8 \\ x^4/4 & x^5/5 & x^6/6 & x^7/7 & x^8/8 & x^9/9 \\ x^5/5 & x^6/6 & x^7/7 & x^8/8 & x^9/9 & x^{10}/10 \\ x^6/6 & x^7/7 & x^8/8 & x^9/9 & x^{10}/10 & x^{11}/11 \end{bmatrix} \quad \mathbf{D}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 4 & 0 & 0 \\ 1 & 0 & -8 & 0 & 8 & 0 \\ 0 & 5 & 0 & -20 & 0 & 16 \end{bmatrix}$$

Moreover, the matrix form for conditions can be written as:

$$\begin{bmatrix} y_0(0) \\ y_1(0) \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{A}^0 \\ \mathbf{A}^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (28)$$

$$\begin{bmatrix} y_0^{'}(0) \\ y_1^{'}(0) \end{bmatrix} \equiv \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 & 5 \end{bmatrix} \begin{bmatrix} \mathbf{A}^0 \\ \mathbf{A}^1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

From system (27) and conditions (28), the new augmented matrix $\overline{\mathbf{W}}$ and $\overset{*}{\mathbf{F}}$ is the systems of linear equations with ten unknowns, then solving this system, Chebyshev coefficients matrix are obtained as:

$$\mathbf{A}^0 = \begin{bmatrix} 1.265647 \\ 1.129553 \\ 0.270745 \\ 0.043790 \\ 0.005098 \\ 0.000363 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^1 = \begin{bmatrix} 1.265253 \\ -1.132344 \\ 0.269956 \\ -0.045529 \\ 0.004702 \\ -0.000848 \end{bmatrix}$$

Thereby, the solutions of the problem for $N = 5$ become

$$y_0^5(t) = 1 + t + 0.500703t^2 + 0.167888t^3 + 0.040788t^4 + 0.005818t^5$$
$$y_1^5(t) = 1 - t + 0.502289t^2 - 0.165140t^3 + 0.037622t^4 - 0.013582t^5.$$

In Tables 1 and 2, we compare the exact solutions and approximate values for various N. Figs. 1 and 2 display the comparison of absolute errors for various N. Moreover, we compare the absolute errors and error estimation function in Fig.3. The numerical results show that the accuracy improves when N is increased. Tables and figures indicate that as N increases the errors decrease; hence for better results, using large number N is recommended.

**Table 1. Numerical result for approximate solution of $y_0(t)$ in Example 1**

t	Exact solution	N=5	N_e=5	N=6	N_e=6	N=7	N_e=7
-1.0	0.367879	0.367784	0.949E-4	0.367830	0.439E-4	0.367879	0.283E-6
-0.8	0.449328	0.449291	0.375E-4	0.449292	0.369E-4	0.449329	0.888E-7
-0.6	0.548811	0.548822	0.113E-4	0.548788	0.235E-4	0.548811	0.116E-6
-0.4	0.670320	0.670352	0.322E-4	0.670310	0.983E-5	0.670319	0.279E-6
-0.2	0.818730	0.818748	0.176E-4	0.818729	0.142E-5	0.818730	0.170E-6
0.0	1.000000	1.000000	0.000E-0	1.000000	0.000E-0	1.000000	0.000E-0
0.2	1.221402	1.221438	0.356E-4	1.221406	0.362E-5	1.221402	0.303E-6
0.4	1.491824	1.491961	0.136E-3	1.491850	0.253E-4	1.491823	0.763E-6
0.6	1.822118	1.822255	0.136E-3	1.822179	0.607E-4	1.822119	0.228E-6
0.8	2.225540	2.225022	0.518E-3	2.225561	0.204E-3	2.225541	0.886E-6
1.0	2.718281	2.715199	0.308E-2	2.717893	0.388E-3	2.718260	0.213E-4

Table 2. Numerical result for approximate solution of $y_1(t)$ in Example 1

t	Exact solution	N=5	N_e=5	N=6	N_e=6	N=7	N_e=7
-1.0	2.718281	2.718636	0.354E-4	2.718313	0.319E-4	2.718280	0.149E-5
-0.8	2.225540	2.225878	0.337E-4	2.225570	0.299E-4	2.225539	0.133E-5
-0.6	1.822118	1.822426	0.308E-3	1.822142	0.240E-4	1.822117	0.132E-5
-0.4	1.491824	1.492037	0.212E-3	1.491836	0.119E-4	1.491823	0.116E-5
-0.2	1.221402	1.221477	0.745E-4	1.221404	0.202E-5	1.221402	0.493E-6
0.0	1.000000	0.999999	0.200E-9	0.999999	0.000E-0	1.000000	0.000E-0
0.2	0.818730	0.818826	0.955E-3	0.818725	0.490E-5	0.818730	0.593E-6
0.4	0.670320	0.670621	0.301E-3	0.670282	0.372E-4	0.670318	0.111E-5
0.6	0.548811	0.548973	0.162E-3	0.548710	0.101E-3	0.548812	0.135E-5
0.8	0.449328	0.447873	0.145E-2	0.449224	0.104E-3	0.449331	0.221E-5
1.0	0.367879	0.361189	0.668E-2	0.368151	0.271E-3	0.367841	0.376E-4
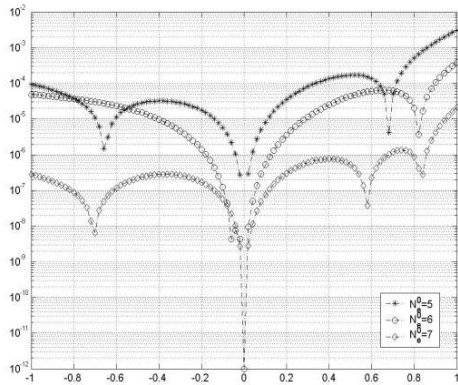
Fig. 1. Comparison of the absolute errors for $y_0(t)$ in Ex. 1
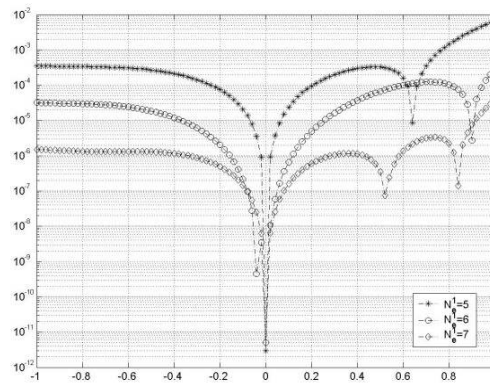
Fig. 2. Comparison of the absolute errors for $y_1(t)$ in Ex. 1

Example 2. Let us consider a system of first-order linear VIDEs

$$y_0'(t) + y_1(t) = 1 + t + t^2 - \int_0^t (y_0(x) + y_1(x))dx$$

$$y_0'(t) - y_1(t) = -1 - t - \int_0^t (y_0(x) - y_1(x))dx$$

and

$$y_0(0) = 1, \ y_1(0) = -1.$$

The exact solutions are $y_0 = t + e^t$, $y_1 = t - e^t$. Table 3 contains a numerical comparision of absolute errors between our solution Chebyshev collocation method and the solutions obtained by HPM [12] and ADM [44]. Present method is incisive, because for the same number basis functions it obtains beter results. Moreover, we give the comparison of absolute errors for present methods for various N in Figs. 3 and 4, and error estimation function display in Fig. 5.

**Table 3. The values of seven terms approximations with related errors**

t	Absolute errors of ADM		Absolute errors of HPM		Present method	
	$N_e^0 = 7$	$N_e^1 = 7$	$N_e^0 = 7$	$N_e^1 = 7$	$N_e^0 = 7$	$N_e^1 = 7$
0.0	0.000E-0	0.000E-0	0.000E-0	0.000E-0	0.100E-9	0.100E-9
0.2	0.300E-8	0.200E-8	0.300E-8	0.200E-8	0.466E-7	0.404E-7
0.4	0.320E-6	0.320E-6	0.320E-6	0.320E-6	0.130E-6	0.747E-7
0.6	0.536E-5	0.535E-5	0.536E-5	0.535E-5	0.700E-6	0.824E-6
0.8	0.390E-4	0.390E-4	0.390E-4	0.390E-4	0.246E-8	0.273E-6
1.0	0.179E-3	0.179E-3	0.179E-3	0.179E-3	0.873E-6	0.107E-5
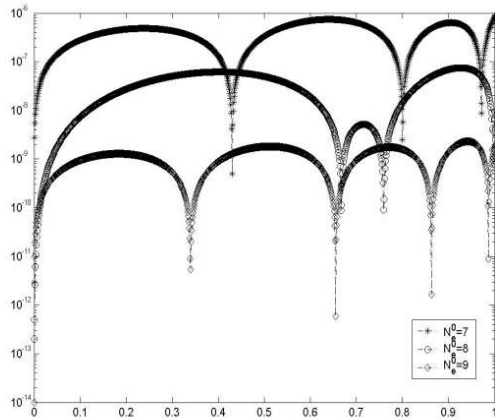
Fig. 3. Comparison of the absolute errors for $y_0(t)$ in Ex. 2
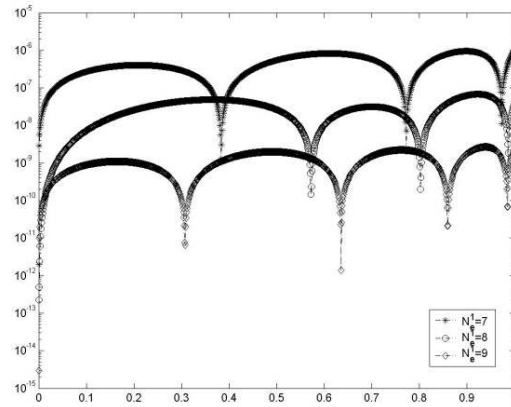
Fig. 4. Comparison of the absolute errors for $y_1(t)$ in Ex. 2
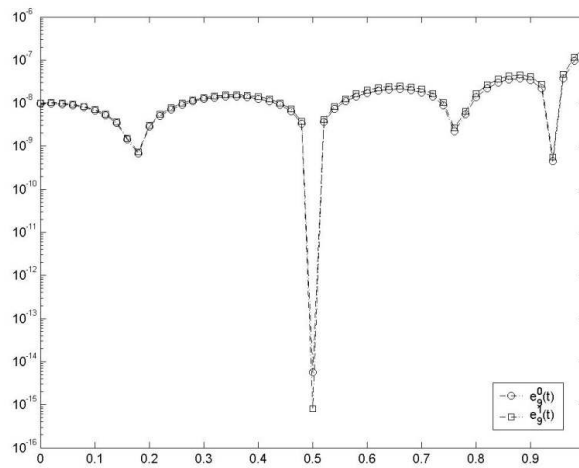
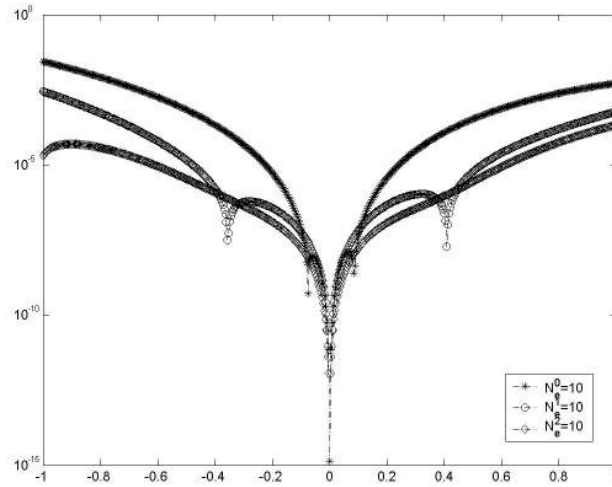**Fig. 5. Plot of the error estimation function for N=9 in Ex. 2**

Fig. 6. Plot of the absolute error in Ex. 3

Example 3. Now, we consider the following problem

$$y_0^{''}(t) - ty_1^{''}(t) + y_2^{'}(t) - 2ty_0(t) - y_1(t) = \int_0^x \left(txy_0(x) + (x-t)y_1(x) \right)dx + f_0(t)$$

$$y_1^{''}(t) - \cos(t)y_2^{''}(t) + \sin(t)y_0^{'}(t) - ty_2(t) = \int_0^x \left(xy_2(x) - e^t y_0(x) \right)dx + f_1(t)$$

$$y_2^{''}(t) - (t-1)y_1^{'}(t) - y_0(t) + \cos(t)y_1(t) = \int_0^x \left(ty_1(x) - \cos(t)y_2(x) \right)dx + f_2(t)$$

with

$$y_0(0) = 0, \ y_1(0) = 1, \ y_2(0) = -1, \ y_0^{'}(0) = 1, \ y_1^{'}(0) = 0, \ y_2^{'}(0) = 1.$$

where

$$f_0(t) = t^2 \cos(t) + t\cos(t) - \cos(t) - 3t\sin(t) + 1$$
$$f_1(t) = 2\sin(t)\cos(t) - \cos^2(t) + 2t\cos(t) - 1 + e^t - \sin(t) - e^t \sin(t)$$
$$f_2(t) = 2\cos^2(t) - 3\sin(t) + \sin(t)\cos(t)$$

The exact solution of this system is $y_0(t) = \sin(t)$, $y_1(t) = \cos(t)$, $y_2(t) = \sin(t) - \cos(t)$. In Fig. 6, we display the absolute error obtained by present method for $N = 10$.

Example 4. Let us consider the following systems

$$y_0^{'}(t) + e^{-t}y_1(t) - y_0(t/2) + e^{t/2}y_1^{'}(t/2) = \int_0^t (x-t)^2 y_0(x)dx + f_0(t)$$

$$y_0^{'}(t) - e^{-t}y_1^{'}(t) - y_0(t) + e^{t/2}y_1(t/2) - ty_0^{'}(t/2) = \int_0^t (t-x)^2 y_1(x)dx + f_1(t)$$

with initial condition $y_0(0) = 0$, $y_1(0) = 0$ and

$$f_0(t) = (7-t)e^t + te^{-t^2} + \frac{1}{2}e^{t/2}(-t - \frac{1}{4} + e^{-t/2} - te^{t/4}) - t^2 - 4t - 6,$$

$$f_1(t) = (1-t)e^{-t^2} + e^t + \frac{1}{2}e^{t/2}(te^{-t/2} - t - \frac{1}{4}t^2) + e^{-t}(6+2t) - t^2 + 4t - 6.$$

The exact solution of this system is $y_0(t) = te^{-t}$, $y_1(t) = te^t$. The computational results, which are obtained by present method are given nin Table 4 and 5. Figs. 11 and 12 display comparison of the absolute errors for various N.

Table 4. Numerical result for approximate solution of $y_0(t)$ in Example 4

t	Exact solution	N=8	N_e=8	N=9	N_e=9	N=10	N_e=10
-1.0	-0.368794	-0.368792	0.161E-6	-0.368794	0.148E-7	-0.368794	0.414E-9
-0.8	-0.359463	-0.359463	0.393E-7	-0.359463	0.738E-8	-0.359463	0.142E-9
-0.6	-0.329286	-0.329286	0.730E-7	-0.329286	0.435E-8	-0.329286	0.252E-9
-0.4	-0.268128	-0.268127	0.246E-6	-0.268128	0.498E-9	-0.268128	0.406E-9
-0.2	-0.163746	-0.163746	0.154E-6	-0.163746	0.131E-7	-0.163746	0.559E-9
0.2	0.244280	0.244280	0.192E-6	0.244280	0.209E-7	0.244280	0.685E-9
0.4	0.596729	0.596729	0.300E-6	0.596729	0.473E-8	0.596729	0.267E-9
0.6	1.093271	1.093271	0.245E-6	1.093271	0.375E-8	1.093271	0.866E-9
0.8	1.780432	1.780432	0.170E-7	1.780432	0.641E-8	1.780432	0.239E-8
1.0	2.718281	2.718281	0.107E-6	2.718281	0.851E-8	2.718281	0.361E-8

Table 5. Numerical result for approximate solution of $y_1(t)$ in Example 4

t	Exact solution	N=8	N_e=8	N=9	N_e=9	N=10	N_e=10
-1.0	-2.718281	-2.718281	0.190E-6	-2.718281	0.1068E-7	-2.718281	0.135E-5
-0.8	-1.780432	-1.780432	0.148E-6	-1.780432	0.842E-8	-1.780432	0.288E-9
-0.6	-1.093271	-1.093271	0.200E-6	-1.093271	0.535E-8	-1.093271	0.346E-10
-0.4	-0.596729	-0.596729	0.373E-6	-0.596729	0.998E-8	-0.596729	0.744E-9
-0.2	-0.244280	-0.244280	0.200E-6	-0.244280	0.631E-8	-0.244280	0.714E-9
0.2	0.163746	0.163746	0.193E-6	0.163746	0.809E-8	0.163746	0.666E-9
0.4	0.268128	0.268128	0.465E-6	0.268128	0.173E-7	0.268128	0.145E-9
0.6	0.329286	0.329286	0.339E-6	0.329286	0.175E-8	0.329286	0.809E-9
0.8	0.359463	0.359463	0.321E-6	0.359463	0.341E-7	0.359463	0.457E-8
1.0	0.367879	0.367879	0.117E-5	0.367879	0.251E-8	0.367879	0.809E-8

Table 6. Compare of some numerical values for Examples 2 and 4

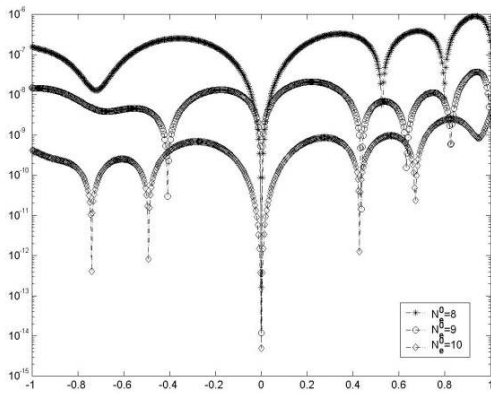|	$N = 8$	
	Example 2	**Example 4**
$\max\left|y_0(t) - y_0^N(t)\right|$	0.931160E-8	0.159960E-12
$\max\left|y_1(t) - y_1^N(t)\right|$	0.517564E-7	0.132420E-12
$E_L^0(N)$	0.456041E-5	0.509900E-6
$E_L^1(N)$	0.213654E-4	0.111080E-5
$E_T^0(N)$	0.993247E-8	0.891350E-7
$E_T^1(N)$	0.981129E-8	0.113042E-6

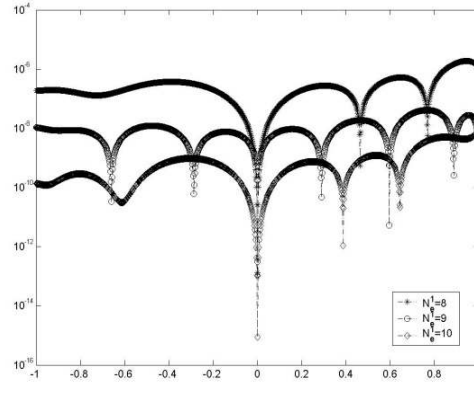**Fig. 7. Comparison of the absolute errors for $y_0(t)$ in Ex. 4**

Fig. 8. Comparison of the absolute errors for $y_1(t)$ in Ex. 4

5. CONCLUSION

In the present paper we used Chebyshev collocation method to solve systems of Volterra-differential-difference equations. The main idea of the proposed method is to convert the problem including linear algebraic equation and find the Chebyshev coefficients in truncated Chebyshev sum. Numerical examples reveal that the present method is very accurate and convenient for solving systems of high order linear VFIDEs. Tables and figures indicate that as N increases, the errors decrease more rapidly; hence for better results, using large number N is recommended. We compare the some computational errors for Examples 2 and 4 suchs as maximum errros, truncation errors and L^2-norm errors in Table 6.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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