Traveling Wave Solutions of the Nonlinear (1+1)-Dimensional Modified Benjamin-Bona-Mahony Equation by Using Novel (G'/G)-Expansion Method

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Authors' contributions

This work was carried out in collaboration between the authors. Both authors had a good contribution to design the study, and to perform the analysis of this research work. Both authors read and approved the final manuscript.

Research Article

ABSTRACT

Exact solutions of nonlinear evolution equations play very important role to make known the inner mechanism of intricate physical phenomena. In this article, the novel (G'/G)-expansion method is applied to construct traveling wave solutions of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation. The performance of this method is reliable, effective and giving many new exact solutions than the existing methods. The obtained solutions are expressed in terms of hyperbolic, trigonometric and rational functions including solitary and periodic solutions which have many potential applications in physical science and engineering.

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1. INTRODUCTION

Nonlinear evolution equations (NLEEs) have many important applications in several aspects of mathematical-physical sciences as well as other natural and applied sciences. Essentially all the fundamental equations of physics are nonlinear and in general such NLEEs are often very difficult to solve explicitly. The exact solutions of NLEEs play an important role in the study of nonlinear physical phenomena. Therefore, the powerful and efficient methods to find exact solutions of nonlinear equations still have drawn a lot of interest by a diverse group of scientists. In the past decades, there has been significant progress in the development of finding effective methods for obtaining exact solutions of NLEEs. With the invention of symbolic computation software, like-Maple or Mathematica direct methods to search for exact solutions of NLEEs have attracted more attention. As a result, the researchers developed and established many methods, for example, the Exp-function method [1-4], the inverse scattering transform [5], the sine-cosine method [6], the extended tanh-method [7], the parameter-expansion method [8], the homogeneous balance method [9], the Backlund transform method [10], the Darboux transformation [11], the Hirota bilinear method [12], the symmetry method [13], the Painleve expansion [14], the (G'/G)-expansion method [15-26], the Cole-Hopf transformation [27], the modified simple equation method [28-32], the improved (G'/G)-expansion method [33,34] and so on to construct exact solution of NLEEs.

Recently, Alam et al. [35] established a highly effective extension of the (G'/G)-expansion method, called the novel (G'/G)-expansion method to obtain exact traveling wave solutions of NLEEs. The objective of this article is to present an application relating to the new (G'/G)-expansion method to find hyperbolic, trigonometric and rational functions solutions of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation to demonstrate the suitability and straightforwardness of the method.

The foremost advantage of the method applied in this article over the basic (G'/G)-expansion method is that it provides further new exact traveling wave solutions including additional free parameters. All the solutions obtained by the basic (G'/G)-expansion method are obtained through the applied method as a particular case and we obtain some new solutions as well. The exact solutions have its great importance to uncover the inner mechanism of the physical phenomena. Apart from the physical relevance, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

In the basic (G'/G)-expansion method, if the order of the reduced ordinary differential equation (ODE) is less than or equal to three, with the help of computer algebra, such as Maple 13, it is mostly possible to find out a useful solution of the algebraic equations resulted in step 4 of section 2. Otherwise, it is generally unable to guarantee the existence of a solution of the resulted algebraic equations; this is because the number of the equations included in the set of algebraic equations is generally greater than the number of unknowns. But the applied method might be used less than or equal to fourth order reduced ODE, since it contains further arbitrary constants compared to the basic (G'/G)-expansion method.

The rest of the article is organized as follows: In Section 2, the description of the novel (G'/G)-expansion method and Remark 1 are given. In Section 3, we apply this method to the (1+1)-dimensional modified Benjamin-Bona-Mahony equation to obtain the traveling wave

solution. In Sections 4 and 5, we give some discussions and physical explanation and in Sections 6, conclusions are given.

2. MATERIALS AND METHOD

Suppose the nonlinear evolution equation is of the form

$$P(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \cdots) = 0 , \tag{1}$$

where P is a polynomial in $u(x,t)$ and its partial derivatives wherein the highest order partial derivatives and the nonlinear terms are involved. The main steps of the method are as follows:

Step 1: Combining the real variables x and t by a compound variable ξ, we suppose that

$$u(x,t) = u(\xi), \ \ \xi = x \pm V t , \tag{2}$$

where V is the speed of the traveling wave. Eq. (2) transforms Eq. (1) into an ODE for $u = u(\xi)$:

$$Q(u, u', u'', u''', \cdots) = 0 , \tag{3}$$

where Q is a function of $u(\xi)$ and its derivatives wherein prime stands for derivative with respect to ξ.

Step 2: Assume the solution of Eq. (3) can be expressed in powers $\psi(\xi)$:

$$u(\xi) = \sum_{j=-N}^{N} \alpha_j \big(\psi(\xi)\big)^j \tag{4}$$

where

$$\psi(\xi) = \big(d + \Phi(\xi)\big) \tag{5}$$

and $\quad \Phi(\xi) = \dfrac{G'(\xi)}{G(\xi)} .$

Herein α_{-N} or α_N may be zero, but both of them could not be zero simultaneously. α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$ and d are constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ODE (see [36] for details):

$$G G'' = \lambda G G' + \mu G^2 + \upsilon (G')^2 \tag{6}$$

where prime denotes the derivative with respect ξ; λ, μ, and υ are real parameters.

The Cole-Hopf transformation $\Phi(\xi) = \ln(G(\xi))_\xi = \dfrac{G'(\xi)}{G(\xi)}$ reduces the Eq. (6) into Riccati equation:

$$\mathbf{\Phi}'(\xi) = \mu + \lambda\,\mathbf{\Phi}(\xi) + (\upsilon - 1)\,\mathbf{\Phi}^2(\xi) \tag{7}$$

Eq. (7) has individual twenty five solutions ([37] for details).

Step 3: The value of the positive integer N can be determined by balancing the highest order linear terms with the nonlinear terms of the highest order come out in Eq. (3). If the degree of $u(\xi)$ is $D[u(\xi)] = n$, then the degree of the other expressions will be as follows:

$$D[\frac{d^p u(\xi)}{d\xi^p}] = n + p, \quad D[u^p\left(\frac{d^q u(\xi)}{d\xi^q}\right)^s] = n\,p + s\,(n+q).$$

Step 4: Substitute Eq. (4) including Eqs. (5) and (6) into Eq. (3), we obtain polynomials in $\left(d + \dfrac{G'(\xi)}{G(\xi)}\right)^j$ and $\left(d + \dfrac{G'(\xi)}{G(\xi)}\right)^{-j}$, $(j = 0, 1, 2, \cdots, N)$. Collect each coefficient of the resulted polynomials to zero, yields an over-determined set of algebraic equations for α_j $(j = 0, \pm 1, \pm 2, \cdots, \pm N)$, d and V.

Step 5: Suppose the value of the constants can be obtained by solving the algebraic equations obtained in Step 4. Substituting the values of the constants together with the solutions of Eq. (6), we will obtain new and comprehensive exact traveling wave solutions of the nonlinear evolution equation (1).

Remark 1: It is noteworthy to observe that if we replace λ by $-\lambda$ and μ by $-\mu$ and put $\upsilon = 0$ in Eq. (6), then the applied novel (G'/G)-expansion method coincide with Akbar et al.'s [17] generalized and improved (G'/G)-expansion method. On the other hand, if we put $d = 0$ in Eq. (5) and $\upsilon = 0$ in Eq. (6) then the proposed method is identical to the improved (G'/G)-expansion method presented by Zhang et al. [33]. Again if we set $d = 0$, $\upsilon = 0$ and the negative exponents of (G'/G) are zero in Eq. (4), then the proposed method turn into the basic (G'/G)-expansion method introduced by Wang et al. [15]. Finally, if we put $\upsilon = 0$ in Eq. (6) and α_j $(j = 1, 2, 3, \cdots, N)$ are functions of x and t instead of constants then the proposed method is transformed into the generalized the (G'/G)-expansion method developed by Zhang et al. [19]. Thus the methods presented in the Ref. [15, 17, 19, 33] are only special cases of the applied novel (G'/G)-expansion method.

3. APPLICATION

In this section, we will bring to bear the new (G'/G) expansion method to construct new and more general traveling wave solutions of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation. The equation was introduced by Benjamin, Bona, and Mahony in 1972 as an improvement of the KdV equation for modeling long waves of small amplitude in (1+1)-dimensions surface water waves propagating uni-directionally and suffering nonlinear and dispersive effects. They showed the stability and uniqueness of solutions to the BBM equation. In contrasts with the KdV equation this equation is unstable in its high wave number components. Let us consider the (1+1)-dimensional modified Benjamin-Bona-Mahony equation.

$$u_t + u_x - \alpha u^2 u_x + u_{xxx} = 0 . \tag{8}$$

Using the traveling wave transformation $\xi = x - V t$, Eq. (8) is converted into the following ODE:

$$(1-V)u' - \alpha u^2 u' + u''' = 0 . \tag{9}$$

Integrating Eq. (9), we obtain

$$K + (1-V)u - \frac{1}{3}\alpha u^3 + u'' = 0 \tag{10}$$

where K is an integration constant. Considering the homogeneous balance between the highest-order derivative u'' and nonlinear term of the highest order u^3 in Eq. (10), we obtain $N = 1$.

Therefore, the solution of Eq. (10) takes the form

$$u(\xi) = \alpha_{-1}\left(\psi(\xi)\right)^{-1} + \alpha_0 + \alpha_1\left(\psi(\xi)\right). \tag{11}$$

Inserting Eq. (11) into Eq. (10), the left hand side is transformed into polynomials in $\left(d + \dfrac{G'(\xi)}{G(\xi)}\right)$ and $\left(d + \dfrac{G'(\xi)}{G(\xi)}\right)^{-1}$. Equating the coefficients of like power of these polynomials to zero, we obtain a set of algebraic equations (for minimalism we leave out to display the equations) for α_0, α_1, α_{-1}, d, K and V. Solving the over-determined set of algebraic equations by using the symbolic computation software, such as, Maple 13, we obtain

Set 1: $K = 0$, $\alpha_0 = \pm\dfrac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}}$, $V = 2\mu\upsilon + 1 - 2\mu - \dfrac{1}{2}\lambda^2$,

$$\alpha_{-1} = 0, \quad d = d, \quad \alpha_1 = \pm\sqrt{\frac{6}{\alpha}}(\upsilon - 1). \tag{12}$$

Set 2: $K = 0$, $\alpha_0 = \pm\dfrac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}}$, $\alpha_{-1} = \pm\sqrt{\dfrac{6}{\alpha}}(\mu - \lambda d + \upsilon d^2 - d^2)$,

$$V = 2\mu\upsilon + 1 - 2\mu - \frac{1}{2}\lambda^2, \quad d = d, \quad \alpha_1 = 0, \tag{13}$$

where d, λ, μ, α and υ are arbitrary constants.

Substituting (12)-(13) into solution Eq. (11), we obtain

$$u_1(x,t) = \pm\frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}} \pm \sqrt{\frac{6}{\alpha}}(\upsilon - 1)\big(d + (G'/G)\big) \tag{14}$$

where $\xi = x - \left(2\mu\upsilon + 1 - 2\mu - \dfrac{1}{2}\right)t$, and α, d, λ, μ and υ are arbitrary constants.

$$u_2(x,t) = \pm\frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}} \pm \sqrt{\frac{6}{\alpha}}(\mu - \lambda d + \upsilon d^2 - d^2)\big(d + (G'/G)\big)^{-1} \tag{15}$$

where $\xi = x - \left(2\mu\upsilon + 1 - 2\mu - \dfrac{1}{2}\right)t$, and α, d, λ, μ and υ are arbitrary constants.

Substituting the value of (G'/G) into Eq. (14) and simplifying, we achieve the following solutions:

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu > 0$ and $\lambda(\upsilon - 1) \neq 0$ (or $\mu(\upsilon - 1) \neq 0$),

$$\begin{aligned} u_{1_1}(x,t) = &\pm\sqrt{\frac{6}{\alpha}}(\upsilon - 1) \times \left\{ d - \frac{1}{2(\upsilon - 1)}\left(\lambda + \sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi)\right)\right\} \\ &\pm\frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}}. \end{aligned} \tag{16}$$

where $\xi = x - \left\{2\mu\upsilon + 1 - 2\mu - \dfrac{\lambda^2}{2}\right\}t$, d, α and λ, μ and υ are arbitrary constants.

$$u_{1_2}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1) \times \left\{ d - \frac{1}{2(\upsilon-1)}\left(\lambda + \sqrt{\Omega}\coth(\frac{1}{2}\sqrt{\Omega}\,\xi) \right) \right\}$$

$$\pm\frac{3(-2d+2\upsilon d - \lambda)}{\sqrt{6\alpha}}.$$

17)

$$u_{1_3}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1) \times \left[d - \frac{1}{2(\upsilon-1)}\left\{ \lambda + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}\,\xi) \pm i\sec h(\sqrt{\Omega}\,\xi) \right) \right\} \right]$$

$$\pm\frac{3(-2d+2\upsilon d - \lambda)}{\sqrt{6\alpha}}.$$

(18)

$$u_{1_4}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1) \times \left[d - \frac{1}{2(\upsilon-1)}\left\{ \lambda + \sqrt{\Omega}\left(\coth(\sqrt{\Omega}\,\xi) \pm \csc h(\sqrt{\Omega}\,\xi) \right) \right\} \right]$$

$$\pm\frac{3(-2d+2\upsilon d - \lambda)}{\sqrt{6\alpha}}.$$

(19)

$$u_{1_5}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times \left[d - \frac{1}{4(v-1)}\left\{ 2\lambda + \sqrt{\Omega}\left(\tanh(\frac{1}{4}\sqrt{\Omega}\,\xi) + \coth(\frac{1}{4}\sqrt{\Omega}\,\xi) \right) \right\} \right]$$

$$\pm\frac{3(-2d+2\upsilon d - \lambda)}{\sqrt{6\alpha}}.$$

(20)

$$u_{1_6}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times \left[d + \frac{1}{2(v-1)}\left\{ -\lambda + \frac{\pm\sqrt{\Omega(A^2+B^2)} - A\sqrt{\Omega}\,\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B} \right\} \right]$$

$$\pm\frac{3(-2d+2\upsilon d - \lambda)}{\sqrt{6\alpha}}.$$

(21)

$$u_{1_7}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times\left[d + \frac{1}{2(\upsilon-1)}\left\{ -\lambda + \frac{\pm\sqrt{\Omega(A^2+B^2)} + A\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi)}{A\sinh(\sqrt{\Omega}\,\xi) + B}\right\}\right] \quad (22)$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

where A and B are real constants.

$$u_{1_8}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{ d + \frac{2\mu\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi)}\right\} \quad (23)$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

$$u_{1_9}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{ d + \frac{2\mu\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\frac{1}{2}\sqrt{\Omega}\,\xi) - \lambda\sinh(\frac{1}{2}\sqrt{\Omega}\,\xi)}\right\} \quad (24)$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

$$u_{1_{10}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{ d + \frac{2\mu\cosh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\sinh(\sqrt{\Omega}\,\xi) - \lambda\cosh(\sqrt{\Omega}\,\xi) \pm i\sqrt{\Omega}}\right\} \quad (25)$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

$$u_{1_{11}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{ d + \frac{2\mu\sinh(\sqrt{\Omega}\,\xi)}{\sqrt{\Omega}\cosh(\sqrt{\Omega}\,\xi) - \lambda\sinh(\sqrt{\Omega}\,\xi) \pm \sqrt{\Omega}}\right\} \quad (26)$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

When $\Omega = \lambda^2 - 4\mu\upsilon + 4\mu < 0$ and $\lambda(\upsilon-1) \neq 0$ (or $\mu(\upsilon-1) \neq 0$),

$$u_{1_{12}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d+\frac{1}{2(\upsilon-1)}\left(-\lambda+\sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(27)

$$u_{1_{13}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d-\frac{1}{2(\upsilon-1)}\left(\lambda+\sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(28)

$$u_{1_{14}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}\,\xi)\pm\sec(\sqrt{-\Omega}\,\xi)\right)\right\}\right]$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(29)

$$u_{1_{15}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left[d-\frac{1}{2(\upsilon-1)}\left\{\lambda+\sqrt{-\Omega}\left(\cot(\sqrt{-\Omega}\,\xi)\pm\csc(\sqrt{-\Omega}\,\xi)\right)\right\}\right]$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(30)

$$u_{1_{16}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times\left[d+\frac{1}{4(\upsilon-1)}\left\{-2\lambda+\sqrt{-\Omega}\left(\tan(\frac{1}{4}\sqrt{-\Omega}\,\xi)-\cot(\frac{1}{4}\sqrt{-\Omega}\,\xi)\right)\right\}\right]$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(31)

$$u_{1_{17}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}-A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\,\xi)}{A\sin(\sqrt{-\Omega}\,\xi)+B}\right\}\right]$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(32)

$$u_{1_{18}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)$$

$$\times\left[d+\frac{1}{2(\upsilon-1)}\left\{-\lambda+\frac{\pm\sqrt{-\Omega(A^2-B^2)}+A\sqrt{-\Omega}\cos(\sqrt{-\Omega}\,\xi)}{A\sin(\sqrt{-\Omega}\,\xi)+B}\right\}\right]$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(33)

where A and B are arbitrary constants such that $A^2-B^2>0$.

$$u_{1_{19}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d-\frac{2\mu\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)+\lambda\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)}\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(34)

$$u_{1_{20}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d+\frac{2\mu\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)-\lambda\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(35)

$$u_{1_{21}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d-\frac{2\mu\cos(\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\sin(\sqrt{-\Omega}\,\xi)+\lambda\cos(\sqrt{-\Omega}\,\xi)\pm\sqrt{-\Omega}}\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(36)

$$u_{1_{22}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\upsilon-1)\times\left\{d+\frac{2\mu\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)}{\sqrt{-\Omega}\cos(\frac{1}{2}\sqrt{-\Omega}\,\xi)-\lambda\sin(\frac{1}{2}\sqrt{-\Omega}\,\xi)\pm\sqrt{-\Omega}}\right\}$$

$$\pm\frac{3(-2d+2\upsilon d-\lambda)}{\sqrt{6\alpha}}.$$

(37)

When $\mu = 0$ and $\lambda(\upsilon - 1) \neq 0$,

$$u_{1_{23}}(x,t) = \pm \sqrt{\frac{6}{\alpha}}(\upsilon - 1) \times \left\{ d - \frac{\lambda\, k}{(\upsilon - 1)\{k + \cosh(\lambda\,\xi) - \sinh(\lambda\,\xi)\}} \right\}$$

$$\pm \frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}}. \tag{38}$$

$$u_{1_{24}}(x,t) = \pm \sqrt{\frac{6}{\alpha}}(\upsilon - 1) \times \left\{ d - \frac{\lambda\{\cosh(\lambda\,\xi) + \sinh(\lambda\,\xi)\}}{(\upsilon - 1)\{k + \cosh(\lambda\,\xi) + \sinh(\lambda\,\xi)\}} \right\}$$

$$\pm \frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}}. \tag{39}$$

where k is an arbitrary constant.

When $(\upsilon - 1) \neq 0$ and $\lambda = \mu = 0$, the solution of Eq. (8) is

$$u_{1_{25}}(x,t) = \pm \sqrt{\frac{6}{\alpha}}(\upsilon - 1) \times \left\{ d - \frac{1}{(\upsilon - 1)\,\xi + c_1} \right\} \pm \frac{3(-2d + 2\upsilon d - \lambda)}{\sqrt{6\alpha}} \tag{40}$$

where c_1 is an arbitrary constant.

For Set 2, substituting the value of (G'/G) into Eq. (15) and simplifying, we achieve the following solutions:

When $\mathbf{\Omega} = \lambda^2 - 4\mu\upsilon + 4\mu > 0$ and $\lambda(\upsilon-1) \neq 0$ (or $\mu(\upsilon-1) \neq 0$),

$$u_{2_1}(x,t) = \pm \sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2) \times \left\{ d - \frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\,\tanh(\frac{1}{2}\sqrt{\Omega}\,\xi)\right) \right\}^{-1}$$

$$\pm \frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}}. \tag{41}$$

where $\xi = x - \left(2\mu\upsilon + 1 - 2\mu - \frac{1}{2}\right)t$, and α, d, λ, μ and υ are arbitrary constants.

$$u_{2_2}(x,t) = \pm \sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2) \times \left\{ d - \frac{1}{2(v-1)}\left(\lambda + \sqrt{\Omega}\,\coth(\frac{1}{2}\sqrt{\Omega}\,\xi)\right) \right\}^{-1}$$

$$\pm \frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}}. \tag{42}$$

$$u_{2_3}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2)$$

$$\times\left[d - \frac{1}{2(v-1)}\left\{\lambda + \sqrt{\Omega}\left(\tanh(\sqrt{\Omega}\,\xi) \pm i\sec h(\sqrt{\Omega}\,\xi)\right)\right\}\right]^{-1} \quad (43)$$

$$\pm\frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}}.$$

The other families of exact solutions of Eq. (8) are omitted for convenience.

When $\mathbf{\Omega} = \lambda^2 - 4\mu v + 4\mu < 0$ and $\lambda(v-1) \neq 0$ (or $\mu(v-1) \neq 0$),

$$u_{2_{12}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2)$$

$$\times\left\{d + \frac{1}{2(v-1)}\left(-\lambda + \sqrt{-\Omega}\tan(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1} \quad (44)$$

$$\pm\frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}},$$

$$u_{2_{13}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2)$$

$$\times\left\{d - \frac{1}{2(v-1)}\left(\lambda + \sqrt{-\Omega}\cot(\frac{1}{2}\sqrt{-\Omega}\,\xi)\right)\right\}^{-1} \quad (45)$$

$$\pm\frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}}.$$

$$u_{2_{14}}(x,t) = \pm\sqrt{\frac{6}{\alpha}}(\mu - \lambda d + vd^2 - d^2)$$

$$\times\left\{d + \frac{1}{2(v-1)}\left\{-\lambda + \sqrt{-\Omega}\left(\tan(\sqrt{-\Omega}\,\xi) \pm \sec(\sqrt{-\Omega}\,\xi)\right)\right\}\right\}^{-1} \quad (46)$$

$$\pm\frac{3(-2d + 2vd - \lambda)}{\sqrt{6\alpha}}.$$

The other families of exact solutions of Eq. (8) are omitted for convenience.
When $(v-1) \neq 0$ and $\lambda = \mu = 0$, the solution of Eq. (8) is

$$u_{2_{25}} = \pm \sqrt{\frac{6}{\alpha}} (\mu - \lambda d + \nu d^2 - d^2) \times \left\{ d - \frac{1}{(\nu - 1)\,\xi + c_1} \right\}^{-1}$$

$$\pm \frac{3(-2d + 2\nu d - \lambda)}{\sqrt{6\alpha}}.$$

(47)

where c_1 is an arbitrary constant.

The other families of exact solutions of Eq. (8) are omitted for convenience.

4. DISCUSSIONS

The advantages and validity of the method over the (G'/G)-expansion method has been discussed in the following:

4.1 Advantages

The crucial advantage of the novel approach against the basic (G'/G)-expansion method is that the method provides more general and large amount of new exact traveling wave solutions with several free parameters in a uniform way. The exact solutions have its great importance to expose the inner mechanism of the physical phenomena. Apart from the physical application, the close-form solutions of nonlinear evolution equations assist the numerical solvers to compare the accuracy of their results and help them in the stability analysis.

4.2 Comparison

In Ref. [38] Manafianheris investigated solutions of the well-established (1+1)-dimensional modified Benjamin-Bona-Mahony equation via the (G'/G)-expansion method wherein he used the linear ordinary differential equation $G'' + \lambda G' + \mu G = 0$ as auxiliary equation and traveling wave solution was presented in the form $u(\xi) = \sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$. It is noteworthy to point out that some of our solutions are coincided with already published results, if parameters taken particular values which authenticate our solutions. The comparison among Manafianheris's solutions [38] and the solutions obtained in this article are given in Table 1:

Table 1. Comparison among the solutions obtained in this article and Manafianheris's [38] solutions

| Solutions obtained in this article	Manafianheris's solutions [38]
(i) If $\upsilon = 0$, $d = 0$ and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively then the solution (16) becomes $$u_{1_1} = \pm\sqrt{\frac{3(\lambda^2 - 4\mu)}{2\alpha}}\tanh(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$	(i) If $C_2 = 0$, and a is replaced by $-\alpha$ then the solution (4.9) becomes $$u_1 = \pm\sqrt{\frac{3(\lambda^2 - 4\mu)}{2\alpha}}\tanh(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$
(ii) If $\upsilon = 0$, $d = 0$ and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively then the solution (17) becomes $$u_{1_2} = \pm\sqrt{\frac{3(\lambda^2 - 4\mu)}{2\alpha}}\coth(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$	(ii) If $C_1 = 0$, and a is replaced by $-\alpha$ then the solution (4.9) becomes $$u_1 = \pm\sqrt{\frac{3(\lambda^2 - 4\mu)}{2\alpha}}\coth(\frac{\sqrt{(\lambda^2 - 4\mu)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$
(iii) If $\upsilon = 0$, $d = 0$ and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively then the solution (27) becomes $$u_{1_{12}} = \pm\sqrt{\frac{3(4\mu - \lambda^2)}{2\alpha}}\tan(\frac{\sqrt{(4\mu - \lambda^2)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$	(iii) If $C_2 = 0$, and a is replaced by $-\alpha$ then the solution (4.10) becomes $$u_2 = \pm\sqrt{\frac{3(4\mu - \lambda^2)}{2\alpha}}\tan(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$
(iv) If $\upsilon = 0$, $d = 0$ and λ and μ are replaced by $-\lambda$ and $-\mu$ respectively then the solution (28) becomes $$u_{1_{13}} = \pm\sqrt{\frac{3(4\mu - \lambda^2)}{2\alpha}}\cot(\frac{\sqrt{(4\mu - \lambda^2)}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$	(iv) If $C_1 = 0$, and a is replaced by $-\alpha$ then the solution (4.10) becomes $$u_2 = \pm\sqrt{\frac{3(4\mu - \lambda^2)}{2\alpha}}\cot(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi)$$ $$\pm\frac{3\lambda}{\sqrt{6\alpha}}$$
(v) If $\upsilon = 0$, $d = 0$ then the solution (40) becomes $$u_{1_{25}} = \pm\sqrt{\frac{6}{\alpha}}\frac{1}{C_1 + x - t} \pm \frac{3\lambda}{\sqrt{6\alpha}}$$	(v) If $\lambda^2 - 4\mu = 0$ and a is replaced by $-\alpha$ then the solution (4.11) becomes $$u_3 = \pm\sqrt{\frac{6}{\alpha}}\frac{1}{C_1 + x - t} \pm \frac{3\lambda}{\sqrt{6\alpha}}$$

Manafianheris [38] did not get any solution, but in this article apart from these solutions we obtain a large number of solutions.

5. PHYSICAL EXPLANATION

Solutions (16), (18), (21), (22), (24)-(26), (42) and (43) represent kink. Kink waves are traveling waves which arise from one asymptotic state to another. The kink solutions are approach to a constant at infinity. Fig. 1 below shows the shape of the exact kink-type solution (16) of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation (8). Other figures are omitted for convenience. Solution (22) is the singular kink solution. The Fig. 2 shows the shape of the exact singular kink-type solution (22) of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation (8). Solutions (17), (19), (20), (23), (28), (31), (40) and (41) are the multiple soliton solution. The Fig. 3 shows the shape of the exact the multiple soliton solution (17) of the (1+1)-dimensional modified Benjamin-Bona-Mahony equation (8). Solutions (27), (29), (30), (32)-(37), (44)-(46) represent the exact periodic traveling wave solutions. Periodic solutions are traveling wave solutions that are periodic such as $\cos(x-t)$. Fig. 4 below shows the periodic solution of $u_{1_{12}}(x,t)$ .Graph of periodic solution (27), for $\lambda=1$, $\mu=1$, $\upsilon=1$, $d=1$, $\alpha=1$ with $-1\le x,t\le1$. For convenience other figures are omitted. Solutions (38) and (39) describe the soliton. Solitons are special kinds of solitary waves. The soliton solution is a specially localized solution, hence $u'(\xi),u''(\xi),u'''(\xi)\to0$ as $\xi\to\pm\infty$, $\xi=x-ct$. Solitons have a remarkable property that it keeps its identity upon interacting with other solitons. Fig. 5 shows the soliton obtained from solution (22).
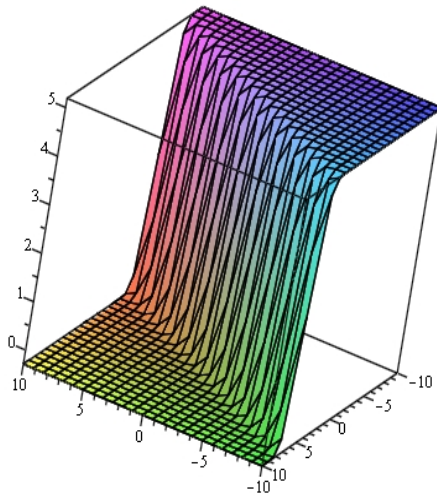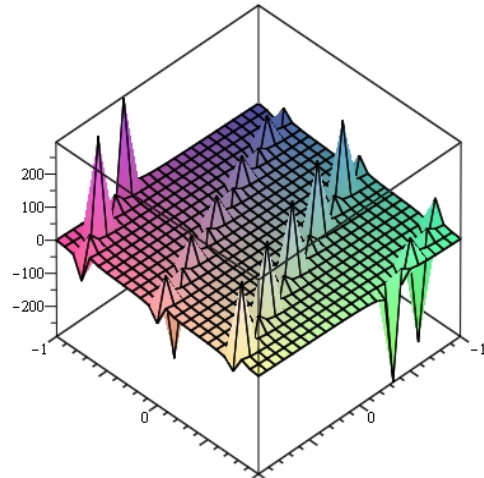

Fig. 1. Kink shape solitary wave obtained from solution (16), for $\lambda=1$, $\mu=-1$, $\upsilon=2$, $d=1$, $\alpha=1$ with $-10\le x,t\le10$.
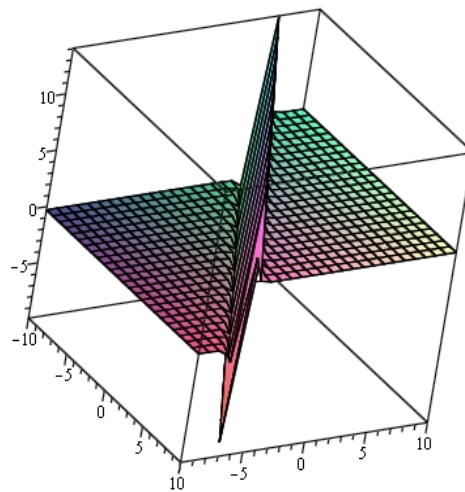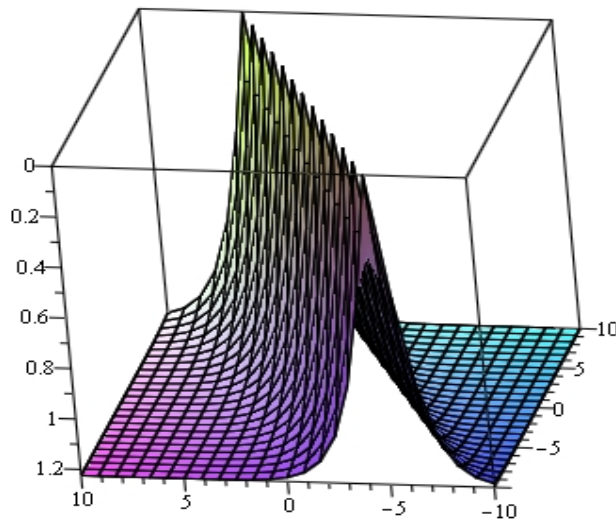
Fig. 2. Singular kink shape solitary wave obtained from solution $u_{1_2}(x,t)$, for $\lambda=1$, $\mu=-1$, $\upsilon=2$, $d=1$, $\alpha=1$ with $-10\le x,t\le10$.
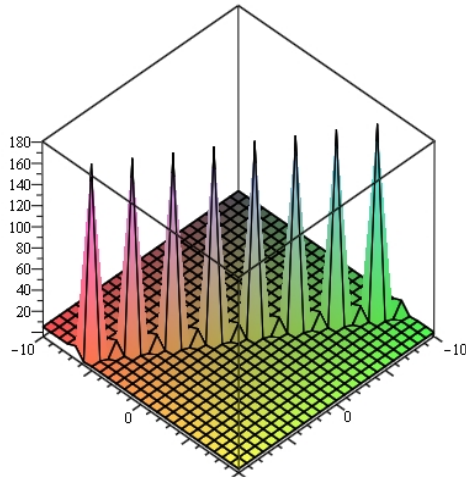
Fig. 3. Multiple soliton obtained from solution (17) for $\lambda = 1$, $\mu = -1$, $\upsilon = 2$, $d = 1$, $\alpha = 1$ **with** $-10 \leq x, t \leq 10$.

Fig. 4. Periodic solution obtained from (27) for $\lambda = 1$, $\mu = 1$, $\upsilon = 2$, $d = 1$, $\alpha = 1$ **with** $-1 \leq x, t \leq 1$.

Fig. 5. Soliton obtained from solution (38) for $\lambda = 1$, $\mu = 0$, $\upsilon = 2$, $d = 1$, $k = 1$, $\alpha = -1$ **with** $-10 \leq x, t \leq 10$.

6. CONCLUSION

The novel (G'/G)-expansion method is successfully applied to establish traveling wave solutions to the (1+1)-dimensional modified Benjamin-Bona-Mahony equation. The performance of this method is reliable, convincing and can be used to other NLEEs in finding exact solutions. The method gives more general solutions which contain further arbitrary constants and the arbitrary constants imply that these solutions have rich local structures. It

is important to notice that the basic (G'/G)-expansion method, the improve (G'/G)-expansion and the generalized and improved (G'/G)-expansion method are only special case of the new (G'/G)-expansion method. It is shown that the novel (G'/G)-expansion method method is straightforward and effective mathematical tool for solving nonlinear evolution equations in mathematical physics and engineering. By means of this scheme, we found some fresh traveling wave solutions of the above mentioned equation. Although the method has a lot of merit it has a few drawbacks, such as, sometimes the method gives solutions in disguised versions of known solutions that may be found by other methods.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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