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Some Applications of Celebrated Master Theorem of Ramanujan

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Abstract

If $F(x)$ is expanded in the form of Maclaurin's series

$$F(x) = \sum_{n=0}^{\infty} \left\{ (-1)^n \frac{\mathrm{d}^n F(x)}{\mathrm{d}x^n} \right\}_{x=0} \frac{(-x)^n}{n!}$$

then Ramanujan asserts that the value of $I = \int_0^\infty x^{s-1} F(x)\,\mathrm{d}x$ can be found from the coefficient of $\frac{(-x)^n}{n!}$ in the expansion of $F(x)$. Conversely Ramanujan claims that if the value of I is known, then the Maclaurin's coefficient of $F(x)$ can be found.

In this paper, we obtain Maclaurin's expansions and Mellin transforms of some composite functions, using *Ramanujan's Master theorem*.

Keywords:Ramanujan Master Theorem; Gauss hypergeometric function; Pochhammer's symbol; Legendre's duplication formula; Mellin transform; Inverse Mellin transform; Maclaurin's expansion; Beta and Gamma functions.

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1 Introduction, Definitions and Preliminaries

Gamma Function is defined by [1,p.347(7.3.1,7.3.2)]

$$\Gamma(s) = \begin{cases} \int_0^\infty e^{-x}\, x^{s-1}\,\mathrm{d}x & ; \text{ if } \Re(s) > 0 \\ \int_0^\infty (e^{-x} - 1)\, x^{s-1}\,\mathrm{d}x\ ; & \text{if } -1 < \Re(s) < 0 \end{cases} \tag{1.1}$$

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Pochhammer's symbol or generalized factorial function $(b)_k$ is defined by [3,p.21(14); see also 2]

$$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = \begin{cases} b(b+1)(b+2)\cdots(b+k-1); & \text{if } k \in \{1,2,3,\dots\} \\ 1 & ; \text{ if } k = 0 \\ k! & ; \text{ if } b = 1,\ k \in \{1,2,3,\dots\} \end{cases} \tag{1.2}$$

The generalized Gaussian hypergeometric function ${}_AF_B$ of one variable is defined by [3,p.42(1); see also 2]

$${}_AF_B\left[\begin{array}{lll} a_1, a_2, \cdots, a_A & ; & z \\ b_1, b_2, \cdots, b_B & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_A)_k z^k}{(b_1)_k (b_2)_k \cdots (b_B)_k k!} \tag{1.3}$$

where denominator parameters $b_1, b_2, \cdots, b_B$ are neither zero nor negative integers and $A,\ B$ are non-negative integers.
If $A \leq B$, then series ${}_AF_B$ is always convergent for all finite values of z(real or complex).
If $A = B + 1$, then series ${}_AF_B$ is convergent for $|z| < 1$.

We recall here one of the gems in Ramanujan's notebooks [4,5,6], which is popularly known as the Ramanujan's Master theorem.

Ramanujan's Master-Theorem [7,p.186;12,p.298(1.1); see also 8]

Suppose that, in some neighbourhood of $x = 0$,

$$F(x) = \sum_{n=0}^{\infty} \frac{\Phi(n)\,(-x)^n}{n!} \tag{1.4}$$

then, Mellin transform of $F(x)$ is given by

$$\mathrm{M}\{F(x) : x \to s\} = \int_0^{\infty} x^{s-1}\, F(x)\, \mathrm{d}x = \Gamma(s)\, \Phi(-s) \tag{1.5}$$

where s is not necessarily a positive integer and $F(x)$ is called inverse Mellin transform of $\Gamma(s)\,\Phi(-s)$ and it is given by

$$\mathrm{M}^{-1}\{\Gamma(s)\,\Phi(-s) : s \to x\} = F(x) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} x^{-s}\, \Gamma(s)\, \Phi(-s)\, \mathrm{d}s \tag{1.6}$$

$$\Big(0 < h < \delta,\ 0 < \delta < 1;\ x > 0\Big)$$

provided that above integral exist.

Mellin transforms[9;10;11] is treated as a sum of two-one sided Laplace transforms.

Above theorems were reported by Ramanujan in his first quarterly report submitted to the university of Madras sometimes in August 1913. There are two other quarterly reports the second and the third submitted later by him. These reports have not been published as such. They are reported in Berndt's note book [12,pp.295-336]. All the three reports contain various applications of Master theorem for the evaluation of a variety of integrals and expansion formulae.

In excellent five "parts"[12,13,14,15,16; see also 17,18] B. C. Berndt has examined 3254 results from the various notebooks left by S. Ramanujan(1887-1920). Also, R. P. Agarwal has made a comprehensive study of Ramanujan's work in his remarkable three "volumes" [7,19,20]. These works also contain complete references of the contributions made by other eminent mathematicians on Ramanujan's mathematics. Undoubtedly, some of Ramanujan's work has embedded in it unparalleled motivation, wisdom, depth, imagination and enough scope to pursue further research.

2 Expansions of Some Functions When Mellin Transforms are Known

[1,p.310(6.2.19)]
Since

$$\int_0^\infty x^{s-1}\,(1+\alpha x)^{-\nu}\,\mathrm{d}x = \alpha^{-s}\,\mathrm{B}(s,\,\nu-s) = \alpha^{-s}\frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)} \tag{2.1}$$

$$\Big(|\arg\alpha|<\pi;\ 0<\Re(s)<\Re(\nu)\Big)$$

then by the application of Ramanujan's Master theorem (1.4) and (1.5), we get

$$(1+\alpha x)^{-\nu} = \sum_{n=0}^{\infty}\alpha^n\,\frac{\Gamma(\nu+n)}{\Gamma(\nu)}\,\frac{(-x)^n}{n!} \tag{2.2}$$

[1,p.311(6.2.28)]
Since

$$\int_0^\infty x^{s-1}\,(a^2+x^2)^{\frac{-1}{2}}\,[x+\sqrt{a^2+x^2}]^{\nu}\,\mathrm{d}x = \frac{a^{\nu+s-1}}{2^s}\,\mathrm{B}\Big(s,\,\frac{1-s-\nu}{2}\Big) = \frac{a^{\nu+s-1}\Gamma(s)\Gamma(\frac{1-s-\nu}{2})}{2^s\Gamma(\frac{1+s-\nu}{2})} \tag{2.3}$$

$$\Big(\Re(a)>0;\ 0<\Re(s)<-\Re(\nu)+1\Big)$$

then

$$\frac{[x+\sqrt{a^2+x^2}]^{\nu}}{\sqrt{a^2+x^2}} = a^{\nu-1}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1+n-\nu}{2})\,(\frac{-2x}{a})^n}{\Gamma(\frac{1-n-\nu}{2})\,n!} \tag{2.4}$$

[1,p.324(6.6.14)]
Since

$$\int_0^\infty x^{s-1}\,\frac{\sinh(\nu\sinh^{-1}x)}{\sqrt{1+x^2}}\mathrm{d}x = \frac{\Gamma(s)\,\sin(\frac{\pi s}{2})\sin(\frac{\pi\nu}{2})\Gamma(\frac{1-s+\nu}{2})\Gamma(\frac{1-s-\nu}{2})}{2^s\,\pi} \tag{2.5}$$

$$\Big(-1<\Re(s)<1-|\Re(\nu)|\Big)$$

then

$$\frac{\sinh(\nu\sinh^{-1}x)}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty}\frac{2^n\,\sin(\frac{-\pi n}{2})\sin(\frac{\pi\nu}{2})\Gamma(\frac{1+n+\nu}{2})\Gamma(\frac{1+n-\nu}{2})}{\pi}\,\frac{(-x)^n}{n!} \tag{2.6}$$

[1,p.324(6.6.15)]
Since

$$\int_0^\infty x^{s-1}\,\frac{\cosh(\nu\sinh^{-1}x)}{\sqrt{1+x^2}}\mathrm{d}x = \frac{\Gamma(s)\,\cos(\frac{\pi s}{2})\cos(\frac{\pi\nu}{2})\Gamma(\frac{1-s+\nu}{2})\Gamma(\frac{1-s-\nu}{2})}{2^s\,\pi} \tag{2.7}$$

$$\Big(0<\Re(s)<1-|\Re(\nu)|\Big)$$

then

$$\frac{\cosh(\nu\sinh^{-1}x)}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty}\frac{2^n\,\cos(\frac{\pi n}{2})\cos(\frac{\pi\nu}{2})\Gamma(\frac{1+n+\nu}{2})\Gamma(\frac{1+n-\nu}{2})}{\pi}\,\frac{(-x)^n}{n!} \tag{2.8}$$

[1,p.348(7.3.9)]
Since

$$\int_0^\infty x^{s-1}\,e^{-x\cos\alpha}\,\sin(x\sin\alpha)\mathrm{d}x = \Gamma(s)\,\sin(\alpha s) \tag{2.9}$$

$$\Big(\Re(s)>-1;\ -\frac{\pi}{2}<\Re(\alpha)<\frac{\pi}{2}\Big)$$

then

$$e^{-x\cos\alpha}\ \sin(x\sin\alpha) = \sum_{n=1}^{\infty}(-1)^{n+1}\ \sin(\alpha n)\ \frac{x^n}{n!} \tag{2.10}$$

[1,p.348(7.3.11)]
Since

$$\int_0^{\infty} x^{s-1}\, e^{-x\cos\alpha}\ \cos(x\sin\alpha)\mathrm{d}x = \Gamma(s)\ \cos(\alpha s) \tag{2.11}$$

$$\Big(\Re(s) > 0;\ -\frac{\pi}{2} < \Re(\alpha) < \frac{\pi}{2}\Big)$$

then

$$e^{-x\cos\alpha}\ \cos(x\sin\alpha) = \sum_{n=0}^{\infty}(-1)^{n}\ \cos(\alpha n)\ \frac{x^n}{n!} \tag{2.12}$$

[1,p.316(6.4.27),p.347(7.3.25)]
Since

$$\int_0^{\infty} x^{s-1}\ \ln(1+2x\cos\theta+x^2)\mathrm{d}x = \frac{2\,\pi\,\mathrm{cosec}\,(\pi \mathrm{s})\cos(\theta \mathrm{s})}{s} \tag{2.13}$$

$$\Big(-1 < \Re(s) < 0;\ -\pi < \theta < \pi\Big)$$

then

$$\ln(1+2x\cos\theta+x^2) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\ \cos(\theta n)\ \frac{x^n}{n} \tag{2.14}$$

$$\Big(|2x\cos\theta+x^2| < 1\Big)$$

[1,p.317(6.4.28)]
Since

$$\int_0^{\infty} x^{s-1}\ \ln(1-2\alpha e^{-x}\cos\theta+\alpha^2 e^{-2x})\mathrm{d}x = -2\,\Gamma(s)\sum_{m=1}^{\infty}\frac{\alpha^m\ \cos(m\theta)}{(m+1)} \tag{2.15}$$

$$\Big(0 < \alpha < 1;\ -\pi < \theta < \pi;\ \Re(s) > 0\Big)$$

then

$$\ln(1-2\alpha e^{-x}\cos\theta+\alpha^2 e^{-2x}) = -2\bigg(\sum_{m=1}^{\infty}\frac{\alpha^m\ \cos(m\theta)}{(m+1)}\bigg)\, e^{-x} \tag{2.16}$$

[1,p.322(6.5.46)]
Since

$$\int_0^{\infty} x^{s-1}\ \tan^{-1}x\,\mathrm{d}x = -\frac{\pi}{2s}\ \sec\Big(\frac{\pi}{2}\Big) \tag{2.17}$$

$$\Big(-1 < \Re(s) < 0\Big)$$

then

$$\tan^{-1}x = \sum_{n=1}^{\infty}(-1)^{n+1}\ \sin\Big(\frac{n\pi}{2}\Big)\ \frac{x^n}{n};\ \ |x| < 1 \tag{2.18}$$

[1,p.336(6.9.3)]
Since

$$\int_0^{\infty} x^{s-1}\, {}_2F_1\left[\begin{array}{ccc}\alpha,\beta & ; & \\ & & -x \\ \gamma & ; & \end{array}\right]\mathrm{d}x = \frac{\mathrm{B}(s,\,\alpha-s)\,\mathrm{B}(s,\,\beta-s)}{\mathrm{B}(s,\,\gamma-s)} \tag{2.19}$$

$$\Big(0 < \Re(s) < \min\{\Re(\alpha),\,\Re(\beta)\}\Big)$$

then

$$ {}_2F_1\left[\begin{array}{lll} \alpha,\beta & ; & \\ & & -x \\ \gamma & ; & \end{array}\right] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\,\Gamma(\beta+n)}{\Gamma(\gamma+n)}\,\frac{(-x)^n}{n!} \tag{2.20} $$

[21,p.285(6.15.2.10)]
Since

$$ \int_0^\infty x^{s-1}\,{}_1F_1\left[\begin{array}{lll} \alpha & ; & \\ & & -x \\ \gamma & ; & \end{array}\right]\mathrm{d}x = \frac{\Gamma(s)\,\Gamma(\gamma)\,\Gamma(\alpha-s)}{\Gamma(\alpha)\,\Gamma(\gamma-s)} \tag{2.21} $$

$$ \Big(0<\Re(s)<\Re(\alpha)\Big) $$

then

$$ {}_1F_1\left[\begin{array}{lll} \alpha & ; & \\ & & -x \\ \gamma & ; & \end{array}\right] = \frac{\Gamma(\gamma)}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\gamma+n)}\,\frac{(-x)^n}{n!} \tag{2.22} $$

[22,p.229(A.8.1.1)]
Since

$$ \int_0^\infty x^{s-1}\,{}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & -kx \\ (b_B) & ; & \end{array}\right]\mathrm{d}x $$

$$ = \frac{\Gamma(b_1)\,\Gamma(b_2)\,\cdots\,\Gamma(b_B)\,\Gamma(a_1-s)\,\Gamma(a_2-s)\,\cdots\,\Gamma(a_A-s)\,\Gamma(s)}{k^s\,\Gamma(a_1)\,\Gamma(a_2)\,\cdots\,\Gamma(a_A)\,\Gamma(b_1-s)\,\Gamma(b_2-s)\,\cdots\,\Gamma(b_B-s)} \tag{2.23} $$

$$ \Big(0<\Re(s)<\min\{\Re(a_j)\},\ j\in\{1,2,3,\ldots,A\}\Big) $$

then

$$ {}_AF_B\left[\begin{array}{lll} (a_A) & ; & \\ & & -kx \\ (b_B) & ; & \end{array}\right] $$

$$ = \frac{\Gamma(b_1)\,\Gamma(b_2)\,\cdots\,\Gamma(b_B)}{\Gamma(a_1)\,\Gamma(a_2)\,\cdots\,\Gamma(a_A)}\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\,\Gamma(a_2+n)\,\cdots\,\Gamma(a_A+n)}{\Gamma(b_1+n)\,\Gamma(b_2+n)\,\cdots\,\Gamma(b_B+n)}\,\frac{(-kx)^n}{n!} \tag{2.24} $$

[11,p.22(Entry 1.2(2.47));23,p.117(17)]
Since

$$ \int_0^\infty x^{s-1}\,[(x+a)+\sqrt{(x^2+2ax)}]^{-\nu}\mathrm{d}x = \frac{2\nu\,a^{-\nu}\,(\frac{a}{2})^s\,\Gamma(2s)\,\Gamma(\nu-s)}{\Gamma(1+\nu+s)} = \frac{\nu\,\Gamma(s)\,\Gamma(\nu-s)\,2^s\,\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,a^{\nu-s}\,\Gamma(\nu+s+1)} \tag{2.25} $$

$$ \Big(0<\Re(s)<\Re(\nu)\Big) $$

then

$$ [(x+a)+\sqrt{(x^2+2ax)}]^{-\nu} = \sum_{n=0}^{\infty}\frac{\nu\,\Gamma(\nu+n)\,2^{-n}\,\Gamma(\frac{1}{2}-n)}{\sqrt{\pi}\,a^{\nu+n}\,\Gamma(\nu+1-n)}\,\frac{(-x)^n}{n!} = a^{-\nu}\,{}_2F_1\left[\begin{array}{lll} \nu,-\nu & ; & \\ & & -\frac{x}{2a} \\ \frac{1}{2} & ; & \end{array}\right] \tag{2.26} $$

[7,p.191(21)]
Since

$$ \int_0^\infty x^{s-1}\left(\frac{2}{1+\sqrt{1+4x}}\right)^{\mu}\mathrm{d}x = \frac{\mu\,\Gamma(s)\,\Gamma(\mu-2s)}{\Gamma(\mu-s+1)} \tag{2.27} $$

$$ \Big(0<\Re(s)<\Re\Big(\frac{\mu}{2}\Big)\Big) $$

then

$$\left(\frac{2}{1+\sqrt{1+4x}}\right)^{\mu} = \mu \sum_{n=0}^{\infty} \frac{\Gamma(\mu+2n)}{\Gamma(\mu+n+1)} \frac{(-x)^n}{n!} = {}_2F_1\left[\begin{array}{ccc} \frac{\mu}{2},\ \frac{\mu+1}{2} & ; & \\ & & -4x \\ \mu+1 & ; & \end{array}\right] \tag{2.28}$$

[1,p.350(7.3.23)]
Since

$$\int_0^{\infty} x^{s-1}\, x^{-\frac{\nu}{2}}\, J_\nu(2\sqrt{x})\,\mathrm{d}x = \frac{\Gamma(s)}{\Gamma(\nu-s+1)} \tag{2.29}$$

$$\Big(0 < \Re(s) < \Re\Big(\frac{\nu}{2}\Big) + \frac{1}{4}\Big)$$

then

$$x^{-\frac{\nu}{2}}\, J_\nu(2\sqrt{x}) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\nu+n+1)} \frac{(-x)^n}{n!} = \frac{1}{\Gamma(\nu+1)}\, {}_0F_1\left[\begin{array}{ccc} \text{——} & ; & \\ & & -x \\ \nu+1 & ; & \end{array}\right] \tag{2.30}$$

[1,p.351(7.3.31)]
Since

$$\int_0^{\infty} x^{s-1}\, \frac{[\sqrt{x}+\sqrt{4+x}]^{1-2\nu}}{2^{1-2\nu}\sqrt{4+x}}\mathrm{d}x = \frac{\Gamma(2s)\,\Gamma(\nu-s)}{\Gamma(\nu+s)} = \frac{\Gamma(s)\, 2^{2s-1}\,\Gamma(s+\frac{1}{2})\,\Gamma(\nu-s)}{\sqrt{\pi}\,\Gamma(\nu+s)} \tag{2.31}$$

$$\Big(0 < \Re(s) < \Re(\nu)\Big)$$

then

$$\frac{[\sqrt{x}+\sqrt{4+x}]^{1-2\nu}}{\sqrt{(4+x)}} = \frac{1}{4^{\nu}}\, {}_2F_1\left[\begin{array}{ccc} \nu,\ 1-\nu & ; & \\ & & -\frac{x}{4} \\ \frac{1}{2} & ; & \end{array}\right] \tag{2.32}$$

[1,p.322(6.5.46)]
Since

$$\int_0^{\infty} x^{s-1}\, \tan^{-1} x\,\mathrm{d}x = -\frac{\pi}{2s}\, \sec\left(\frac{\pi\, s}{2}\right) \tag{2.33}$$

$$\Big(-1 < \Re(s) < 0\Big)$$

then

$$\tan^{-1} x = \sum_{n=1}^{\infty} (-1)^{n+1}\, \sin\left(\frac{n\pi}{2}\right) \frac{x^n}{n}; \quad |x| < 1 \tag{2.34}$$

[1,p.336(6.9.1)]
Since

$$\int_0^{\infty} x^{s-1}\, \exp\left(\frac{-x^2}{4}\right) D_{-\nu}(x)\,\mathrm{d}x = \frac{\sqrt{\pi}\,\Gamma(s)}{2^{\frac{s+\nu}{2}}\,\Gamma(\frac{s+\nu+1}{2})} \tag{2.35}$$

$$\Big(\Re(s) > 0\Big)$$

then

$$\exp\left(\frac{-x^2}{4}\right) D_{-\nu}(x) = \sqrt{\frac{\pi}{2^{\nu}}} \sum_{n=0}^{\infty} \frac{1}{\Gamma(\frac{\nu-n+1}{2})} \frac{(-\sqrt{2}\,x)^n}{n!} \tag{2.36}$$

where $D_{-\nu}(x)$ is parabolic cylinder function.

[1,p.318(6.5.7)]
Since

$$\int_0^{\infty} x^{s-1}\, e^{-\alpha x}\, \sin(\beta x)\mathrm{d}x = \frac{\Gamma(s)\, \sin[s\tan^{-1}(\frac{\beta}{\alpha})]}{(\sqrt{\alpha^2+\beta^2})^s} \tag{2.37}$$

$$\Big(\Re(\alpha) > |\Im(\beta)|;\ \Re(s) > -1\Big)$$

then

$$e^{-\alpha x}\sin(\beta x) = \sum_{n=0}^{\infty}(\alpha^2+\beta^2)^{\frac{n}{2}}\,(-1)^{n+1}\,\sin\Big(n\tan^{-1}\frac{\beta}{\alpha}\Big)\,\frac{x^n}{n!} \tag{2.38}$$

[1,p.320(6.5.27)]
Since

$$\int_0^{\infty} x^{s-1}\,e^{-\alpha x}\cos(\beta x)\mathrm{d}x = \frac{\Gamma(s)\,\cos[s\tan^{-1}(\frac{\beta}{\alpha})]}{(\sqrt{\alpha^2+\beta^2})^s} \tag{2.39}$$

$$\Big(\Re(\alpha) > |\Im(\beta)|;\ \Re(s) > 0\Big)$$

then

$$e^{-\alpha x}\cos(\beta x) = \sum_{n=0}^{\infty}(\alpha^2+\beta^2)^{\frac{n}{2}}\,(-1)^{n}\,\cos\Big(n\tan^{-1}\frac{\beta}{\alpha}\Big)\,\frac{x^n}{n!} \tag{2.40}$$

[1,p.315(6.4.15)]
Since

$$\int_0^{\infty} x^{s-1}\,\ln(1+\alpha x)\,\mathrm{d}x = \frac{\Gamma(s)\,\Gamma(1-s)}{s\,\alpha^s} \tag{2.41}$$

$$\Big(|\arg(\alpha)| < \pi;\ \ -1 < \Re(s) < 0\Big)$$

then

$$\ln(1+\alpha x) = \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{(\alpha x)^n}{n!} \tag{2.42}$$

3 Mellin Transforms of Some Functions When Series Expansions are Known

By means of Leibnitz theorem for successive differentiation, Binomial expansion, decomposition of a fraction into its partial fractions, Demoivre's theorem, Euler's formula, properties of Gamma functions, after long systematic calculation, we can find the Maclaurin's expansion of required functions. Using Ramanujan's Master theorem (1.4) and (1.5), we get Mellin transforms.

Since we know that

$$\sin(ax) = \sum_{n=1}^{\infty}(-a)^n\,\sin\Big(\frac{n\pi}{2}\Big)\,\frac{(-x)^n}{n!} \tag{3.1}$$

then

$$\int_0^{\infty} x^{s-1}\,\sin(ax)\,\mathrm{d}x = \Gamma(s)\,(a)^{-s}\,\sin\Big(\frac{s\pi}{2}\Big) \tag{3.2}$$

Similarly we can obtain

$$\int_0^{\infty} x^{s-1}\,\cos(ax)\,\mathrm{d}x = \Gamma(s)\,(-a)^{-s}\,\cos\Big(\frac{s\pi}{2}\Big)$$

Since

$$\sin^2(ax) = -\frac{1}{2}\sum_{n=0}^{\infty}(-2a)^n\,\cos\Big(\frac{n\pi}{2}\Big)\,\frac{(-x)^n}{n!} \tag{3.3}$$

then

$$\int_0^{\infty} x^{s-1}\,\sin^2(ax)\,\mathrm{d}x = \Gamma(s)\,(-2)^{-s-1}\,a^{-s}\,\cos\Big(\frac{s\pi}{2}\Big) \tag{3.4}$$

$$\Big(-2 < \Re(s) < 0 \Big)$$

Similarly we can obtain

$$\int_0^\infty x^{s-1} \cos^2(ax)\, \mathrm{d}x = \Gamma(s)\,(2)^{-s-1}\,(-a)^{-s} \cos\Big(\frac{s\pi}{2}\Big)$$

Since

$$\frac{1}{(\alpha+x)(\beta+x)} = \frac{1}{(\beta-\alpha)} \sum_{n=0}^{\infty} (\alpha^{-n-1} - \beta^{-n-1})\,(-x)^n \tag{3.5}$$

then

$$\int_0^\infty x^{s-1}\,(\alpha+x)^{-1}(\beta+x)^{-1}\,\mathrm{d}x = \frac{\pi \operatorname{cosec}(\pi s)(\alpha^{s-1} - \beta^{s-1})}{(\beta-\alpha)} \tag{3.6}$$

$$\Big(|\arg(\alpha)| < \pi;\ |\arg(\beta)| < \pi;\ 0 < \Re(s) < 2;\ \alpha \neq \beta \Big)$$

In (3.5) and (3.6), put $\alpha = ae^{i\theta}$ and $\beta = ae^{-i\theta}$, then

$$\int_0^\infty x^{s-1}\,(x^2 + 2ax\cos\theta + a^2)^{-1}\,\mathrm{d}x = -\Gamma(s)\Gamma(1-s)a^{s-2}\operatorname{cosec}\theta \sin[(s-1)\theta] \tag{3.7}$$

$$\Big(a > 0;\ -\pi < \theta < \pi;\ 0 < \Re(s) < 2 \Big)$$

Since

$$\frac{x+\alpha}{(\beta+x)(\gamma+x)} = \sum_{n=0}^{\infty} \Gamma(1+n) \left[\Big(\frac{\beta-\alpha}{\beta-\gamma}\Big)\beta^{-1-n} + \Big(\frac{\gamma-\alpha}{\gamma-\beta}\Big)\gamma^{-1-n} \right] \frac{(-x)^n}{n!} \tag{3.8}$$

then

$$\int_0^\infty \frac{x^{s-1}\,(x+\alpha)}{(\beta+x)(\gamma+x)}\,\mathrm{d}x = \Gamma(s)\Gamma(1-s) \left[\Big(\frac{\beta-\alpha}{\beta-\gamma}\Big)\beta^{s-1} + \Big(\frac{\gamma-\alpha}{\gamma-\beta}\Big)\gamma^{s-1} \right] \tag{3.9}$$

$$\Big(|\arg(\beta)| < \pi;\ |\arg(\gamma)| < \pi;\ 0 < \Re(s) < 1 \Big)$$

Since

$$(x^2 + 2ax\cos\theta + a^2)^{-\nu} = (x + ae^{i\theta})^{-\nu}(x + ae^{-i\theta})^{-\nu} = a^{-2\nu} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\nu)_n\,(\nu)_m\,(-1)^{n+m}\,x^{n+m}}{n!\,m!\,(ae^{i\theta})^n\,(ae^{-i\theta})^m}$$

$$= \sum_{n=0}^{\infty} \frac{(\nu)_n\,a^{-2\nu}}{(ae^{i\theta})^n}\, {}_2F_1 \left[\begin{matrix} -n,\, \nu & ; \\ & & e^{2i\theta} \\ 1-\nu-n & ; \end{matrix} \right] \frac{(-x)^n}{n!} \tag{3.10}$$

then

$$\int_0^\infty x^{s-1}(x^2 + 2ax\cos\theta + a^2)^{-\nu}\,\mathrm{d}x = \frac{\Gamma(s)\Gamma(\nu-s)\,a^{-2\nu}}{\Gamma(\nu)\,(ae^{i\theta})^{-s}}\, {}_2F_1 \left[\begin{matrix} s,\, \nu & ; \\ & & e^{2i\theta} \\ 1-\nu+s & ; \end{matrix} \right] \tag{3.11}$$

$$\Big(-\pi < \theta < \pi \Big)$$

Conclusion

In this paper, we obtain Maclaurin's expansions and Mellin transforms of some composite functions using Ramanujan's Master theorem.

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Competing interests

The authors declare that no competing interests exist.

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