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Type-II generalized Pythagorean bipolar fuzzy soft sets and application for decision making

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Abstract: In the present communication, we introduce the theory of Type-II generalized Pythagorean bipolar fuzzy soft sets and define complementation, union, intersection, AND, and OR. The Type-II generalized Pythagorean bipolar fuzzy soft sets are presented as a generalization of soft sets. We showed De Morgan's laws, associate laws, and distributive laws in Type-II generalized Pythagorean bipolar fuzzy soft set theory. Also, we advocate an algorithm to solve the decision-making problem based on a soft set model.

Keywords: Pythagorean bipolar fuzzy soft set; Type-II generalized Pythagorean bipolar fuzzy soft set; Decision making problem.

MSC: 03E72; 06D72.

1. Introduction

Many uncertain theories put forward as fuzzy set [1], intuitionistic fuzzy set [2], bipolar fuzzy sets [3] and Pythagorean fuzzy set [4]. Zadeh, introduced fuzzy set, suggests that decision-makers solve uncertain problems by considering membership degree. Atanassov introduces the concept of an intuitionistic fuzzy set. It is characterized by a degree of membership and non-membership satisfying the condition that the sum of its membership degree and non-membership degree does not exceed unity. However, we may interact a problem in decision-making events where the sum of the degree of membership and non-membership of a particular attribute is exceeded one-the concept of Pythagorean fuzzy sets introduced by Yager. The theory of soft sets proposed by Molodtsov [5]. It is a tool of parameterization for coping with uncertainties. Compared with other uncertain theories, soft sets reflect the objectivity and complexity of decision-making during actual situations more accurately. It has been an outstanding achievement both in theories and applications.

Moreover, combining soft sets with other mathematical models is also a critical research area. The concept of the fuzzy soft set by Maji [6], intuitionistic fuzzy soft set [7], and Saleem Abdullah *et al.*, initiated the concept of bipolar fuzzy soft sets [8]. Pinaki Majumdara *et al.*, defined the concept of discussed generalized fuzzy soft sets [9]. In recent years, Peng *et al.*, [10] has extended fuzzy soft set to Pythagorean fuzzy soft set. In 2011, Alkhazaleh *et al.*, [11] introduced the concept of possibility fuzzy soft sets. Yager *et al.*, [12] discuss the application for Pythagorean membership grades, complex numbers, and decision making under 2014. In 2018, Mohana *et al.*, [13] interact bipolar Pythagorean fuzzy sets with an application under decision-making problems. Akram *et al.*, [14] initiate the new type of models for decision making based on rough Pythagorean fuzzy bipolar soft set in 2018. Alkhazaleh *et al.*, [15] discussed the theory of generalized interval-valued fuzzy soft set in 2012. Jana and Pal studied bipolar intuitionistic fuzzy soft sets with applications [16]. In 2019, Jana and Pal introduced Pythagorean fuzzy dombi aggregation operators [17]. Recently, Palanikumar *et al.*, [18] discuss the application for possibility Pythagorean bipolar fuzzy soft sets.

This paper aims to extend the concept of generalized fuzzy soft sets to the parameterization of Type-II generalized Pythagorean bipolar fuzzy sets. We shall further establish a similarity measure based on the soft set model.

2. Preliminaries

Definition 1. [13] Let X be a non-empty set of the universe, Pythagorean bipolar fuzzy set(PBFS) A in X is an object having the following form:

$$A = \{x, \zeta_A^+(x), \xi_A^+(x), \zeta_A^-(x), \xi_A^-(x) | x \in X\},$$

where $\zeta_A^+(x), \xi_A^+(x)$, $\zeta_A^-(x), \xi_A^-(x)$ represent the degree of positive membership, degree of positive non-membership, degree of negative membership and degree of negative non-membership of A respectively. Consider the mapping $\zeta_A^+, \xi_A^+ : X \to [0,1]$, $\zeta_A^-, \xi_A^- : X \to [-1,0]$ such that

$$0 \le (\zeta_A^+(x))^2 + (\xi_A^+(x))^2 \le 1 \text{ and } -1 \le -\Big[(\zeta_A^-(x))^2 + (\xi_A^-(x))^2\Big] \le 0.$$

The degree of indeterminacy is determined as

$$\pi_A^+(x) = \Big[\sqrt{1 - (\zeta_A^+(x))^2 - (\xi_A^+(x))^2}\ \Big] \text{ and } \pi_A^-(x) = -\Big[\sqrt{1 - (\zeta_A^-(x))^2 - (\xi_A^-(x))^2}\ \Big].$$

Then $A = \langle \zeta_A^+, \xi_A^+, \zeta_A^-, \xi_A^- \rangle$ is called a Pythagorean bipolar fuzzy number(PBFN).

Definition 2. [13] Given that $\alpha_1 = (\zeta_{\alpha_1}^+, \xi_{\alpha_1}^+, \zeta_{\alpha_1}^-, \xi_{\alpha_1}^-)$, $\alpha_2 = (\zeta_{\alpha_2}^+, \xi_{\alpha_2}^+, \zeta_{\alpha_2}^-, \xi_{\alpha_2}^-)$ and $\alpha_3 = (\zeta_{\alpha_3}^+, \xi_{\alpha_3}^+, \zeta_{\alpha_3}^-, \xi_{\alpha_3}^-)$ are any three PBFN's over (X, E), then the following properties are holds:

1. $\alpha_1^c = (\xi_{\alpha_1}^+, \zeta_{\alpha_1}^+, \xi_{\alpha_1}^-, \zeta_{\alpha_1}^-)$,
2. $\alpha_2 \cup \alpha_3 = \Big[\max(\zeta_{\alpha_2}^+, \zeta_{\alpha_3}^+), \min(\xi_{\alpha_2}^+, \xi_{\alpha_3}^+), \min(\zeta_{\alpha_2}^-, , \zeta_{\alpha_3}^-) \max(\zeta_{\alpha_2}^-, \zeta_{\alpha_3}^-)\Big]$,
3. $\alpha_2 \cap \alpha_3 = \Big[\min(\zeta_{\alpha_2}^+, \zeta_{\alpha_3}^+), \max(\xi_{\alpha_2}^+, \xi_{\alpha_3}^+), \max(\zeta_{\alpha_2}^-, \zeta_{\alpha_3}^-), \min(\xi_{\alpha_2}^-, \xi_{\alpha_3}^-)\Big]$,
4. $\alpha_2 \ge \alpha_3$ if and only if $\zeta_{\alpha_2}^+ \ge \zeta_{\alpha_3}^+$ and $\xi_{\alpha_2}^+ \le \xi_{\alpha_3}^+$ and $\zeta_{\alpha_2}^- \le \zeta_{\alpha_3}^-$ and $\xi_{\alpha_2}^- \ge \xi_{\alpha_3}^-$,
5. $\alpha_2 = \alpha_3$ if and only if $\zeta_{\alpha_2}^+ = \zeta_{\alpha_3}^+$ and $\xi_{\alpha_2}^+ = \xi_{\alpha_3}^+$ and $\zeta_{\alpha_2}^- = \zeta_{\alpha_3}^-$ and $\xi_{\alpha_2}^- = \xi_{\alpha_3}^-$.

Definition 3. [8] Let X be a non-empty set of the universe and E be a set of parameter. The pair $(\mathscr{M}, A)$ is called a bipolar fuzzy soft set(BFSS) on X if $A \subseteq E$ and $\mathscr{M} : A \to \mathscr{GBM}^X$, where $\mathscr{GBM}^X$ is the set of all bipolar fuzzy subsets of X.

Definition 4. [14] Let X be a non-empty set of the universe and E be a set of parameter. The pair $(\mathscr{M}, A)$ is called a Pythagorean bipolar fuzzy soft set(PBFSS) on X if $A \subseteq E$ and $\mathscr{M} : A \to P\mathscr{GBM}^X$, where $P\mathscr{GBM}^X$ is the set of all Pythagorean bipolar fuzzy subsets of X.

Definition 5. [9] Let $X = \{x_1, x_2, ..., x_n\}$ be a non-empty set of the universe and $E = \{e_1, e_2, ..., e_m\}$ be a set of parameter. The pair (X, E) is a soft universe. Consider the mapping $\mathscr{M} : E \to I^X$ and ξ be a fuzzy subset of E, i.e. $\xi : E \to I = [0,1]$, where I^X is the collection of all fuzzy subsets of X. Let $\mathscr{M}_\xi : E \to I^X \times I$ be a function defined as $\mathscr{M}_\xi(e) = (\mathscr{M}(e)(x), \xi(e)), \forall x \in X$. Then $\mathscr{M}_\xi$ is called a generalized fuzzy soft set(GFSS) on (X, E). Here for each parameter e_i, $\mathscr{M}_\xi(e_i) = (\mathscr{M}(e_i)(x), \xi(e_i))$ indicates not only the degree of belongingness of the elements of X in $\mathscr{M}(e_i)$ but also the degree of possibility of such belongingness which is represented by $\xi(e_i)$. So we can write $\mathscr{M}_\xi(e_i)$ as follows:

$$\mathscr{M}_\xi(e_i) = \left(\left\{\frac{x_1}{\mathscr{M}(e_i)(x_1)}, \frac{x_2}{\mathscr{M}(e_i)(x_2)}, ..., \frac{x_n}{\mathscr{M}(e_i)(x_n)}\right\}, \xi(e_i)\right),$$

where $\mathscr{M}(e_i)(x_1)$, $\mathscr{M}(e_i)(x_2)$,...,$\mathscr{M}(e_i)(x_n)$ are the degrees of belongingness and $\xi(e_i)$ is the degree of possibility of such belongingness.

Definition 6. [11] Let $X = \{x_1, x_2, ..., x_n\}$ be a non-empty set of the universe and $E = \{e_1, e_2, ..., e_m\}$ be a set of parameter. The pair (X, E) is a soft universe. Consider the mapping $\mathscr{M} : E \to \mathscr{M}(X)$ and ξ be a fuzzy subset of E, i.e. $\xi : E \to \mathscr{M}(X)$. Let $\mathscr{M}_\xi : E \to \mathscr{M}(X) \times \mathscr{M}(X)$ be a function defined as $\mathscr{M}_\xi(e) = (\mathscr{M}(e)(x), \xi(e)(x)), \forall x \in X$. Then $\mathscr{M}_\xi$ is called a possibility fuzzy soft set(PFSS) on (X, E). Here for each parameter e_i, $\mathscr{M}_\xi(e_i) = (\mathscr{M}(e_i)(x), \xi(e_i)(x))$ indicates not only the degree of belongingness of the elements of X in $\mathscr{M}(e_i)$ but also the degree of possibility of such belongingness which is represented by $\xi(e_i)$. So we can write $\mathscr{M}_\xi(e_i)$ as follows:

$$\mathscr{M}_\xi(e_i) = \left\{ \left(\frac{x_1}{\mathscr{M}(e_i)(x_1)}, \xi(e_i)(x_1) \right), \left(\frac{x_2}{\mathscr{M}(e_i)(x_2)}, \xi(e_i)(x_2) \right), ..., \left(\frac{x_n}{\mathscr{M}(e_i)(x_n)}, \xi(e_i)(x_n) \right) \right\}.$$

3. Type-II generalized Pythagorean bipolar fuzzy soft sets

Definition 7. Let $X = \{x_1, x_2, ..., x_n\}$ be a non-empty set of the universe and $E = \{e_1, e_2, ..., e_m\}$ be a set of parameter. The pair (X, E) is called a soft universe. Suppose that $\mathscr{M} : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X$ and p is a Pythagorean bipolar fuzzy subset of E. That is $p : E \to I$ where I denotes the collection of all Pythagorean bipolar fuzzy subsets of X. If $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I$ is a function defined as

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e) = \langle \mathscr{G}\mathscr{B}\mathscr{M}(e)(x), p(e) \rangle, x \in X,$$

then $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ is a Type-II generalized Pythagorean bipolar fuzzy soft set(Type-II GPBFSS) on (X, E). For each parameter e,

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e) = \left(\left\{ \frac{x_1}{\zeta^+_{\mathscr{M}(e)}(x_1), \xi^+_{\mathscr{M}(e)}(x_1), \zeta^-_{\mathscr{M}(e)}(x_1), \xi^-_{\mathscr{M}(e)}(x_1)}, ..., \frac{x_n}{\zeta^+_{\mathscr{M}(e)}(x_n), \xi^+_{\mathscr{M}(e)}(x_n), \zeta^-_{\mathscr{M}(e)}(x_n), \xi^-_{\mathscr{M}(e)}(x_n)} \right\}, \left(\zeta^+_{p(e)}, \xi^+_{p(e)}, \zeta^-_{p(e)}, \xi^-_{p(e)} \right) \right).$$

Example 1. Let $X = \{x_1, x_2, x_3\}$ be a set of three leptospirosis patients and $E = \{e_1 = \text{high fever}, e_2 = \text{headache}, e_3 = \text{chills}\}$ is a set of parameters. Suppose that $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I$ is given by

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e_1) = \left(\left\{ \begin{array}{c} \frac{x_1}{(0.6,0.7,-0.3,-0.8)} \\ \frac{x_2}{(0.9,0.4,-0.7,-0.5)} \\ \frac{x_3}{(0.8,0.5,-0.2,-0.9)} \end{array} \right\} (0.6, 0.5, -0.8, -0.3) \right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e_2) = \left(\left\{ \begin{array}{c} \frac{x_1}{(0.7,0.4,-0.2,-0.8)} \\ \frac{x_2}{(0.3,0.9,-0.7,-0.4)} \\ \frac{x_3}{(0.5,0.6,-0.2,-0.9)} \end{array} \right\} (0.9, 0.2, -0.7, -0.4) \right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e_3) = \left(\left\{ \begin{array}{c} \frac{x_1}{(0.3,0.7,-0.8,-0.4)} \\ \frac{x_2}{(0.8,0.4,-0.7,-0.3)} \\ \frac{x_3}{(0.9,0.2,-0.5,-0.6)} \end{array} \right\} (0.6, 0.5, -0.7, -0.3) \right).$$

Definition 8. Let X be a non-empty set of the universe and E be a set of parameter. Suppose that $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ are two Type-II GPBFSS's on (X, E). Now $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \subseteq \mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ if and only if

(i) $\mathscr{M}(e)(x) \subseteq \mathscr{N}(e)(x)$ implies

$$\left\{ \begin{array}{l} \zeta^+_{\mathscr{M}(e)}(x) \le \zeta^+_{\mathscr{N}(e)}(x), \;\; \xi^+_{\mathscr{M}(e)}(x) \ge \xi^+_{\mathscr{N}(e)}(x) \\ \zeta^-_{\mathscr{M}(e)}(x) \ge \zeta^-_{\mathscr{N}(e)}(x), \;\; \xi^-_{\mathscr{M}(e)}(x) \le \xi^-_{\mathscr{N}(e)}(x) \end{array} \right\},$$

(ii) $p(e) \subseteq q(e)$ implies

$$\left\{ \begin{array}{l} \zeta^+_{p(e)} \le \zeta^+_{q(e)}, \;\; \xi^+_{p(e)} \ge \xi^+_{q(e)} \\ \zeta^-_{p(e)} \ge \zeta^-_{q(e)}, \;\; \xi^-_{p(e)} \le \xi^-_{q(e)} \end{array} \right\}$$

$\forall e \in E$.

Example 2. Consider the Type-II GPBFSS $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ over (X,E) in Example 1. Let $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ be another Type-II GPBFSS over (X,E) defined as:

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_1) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.7,0.5,-0.6-0.7)} \\ \frac{x_2}{(0.9,0.2,-0.8-0.4)} \\ \frac{x_3}{(0.9,0.1,-0.5-0.8)} \end{array}\right\}(0.6,0.4,-0.9-0.2)\right);$$

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_2) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.8,0.3,-0.4-0.6)} \\ \frac{x_2}{(0.6,0.7,-0.8-0.3)} \\ \frac{x_3}{(0.7,0.4,-0.3-0.7)} \end{array}\right\}(0.9,0.1,-0.8-0.3)\right);$$

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_3) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.5,0.6,-0.9-0.3)} \\ \frac{x_2}{(0.8,0.3,-0.8-0.2)} \\ \frac{x_3}{(0.9,0.1,-0.7-0.5)} \end{array}\right\}(0.7,0.4,-0.9-0.1)\right).$$

Definition 9. Let X be a non-empty set of the universe and E be a set of parameter. Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ be a Type-II GPBFSS on (X,E). The complement of $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ is denoted by $\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}$ and is defined by

$$\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}} = \left\langle \mathscr{G}\mathscr{B}\mathscr{M}^c(e)(x), p^c(e)\right\rangle,$$

where $\mathscr{G}\mathscr{B}\mathscr{M}^c(e)(x) = \left(\xi^+_{\mathscr{M}(e)}(x), \zeta^+_{\mathscr{M}(e)}(x), \xi^-_{\mathscr{M}(e)}(x), \zeta^-_{\mathscr{M}(e)}(x)\right)$ and $p^c(e) = \left(\xi^+_{p(e)}, \zeta^+_{p(e)}, \xi^-_{p(e)}, \zeta^-_{p(e)}\right)$. It is true that $\mathscr{M}_{(p^c)^c}^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$.

Definition 10. Let X be a non-empty set of the universe and E be a set of parameter. Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ be two Type-II GPBFSSs on (X,E). The union and intersection of $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ over (X,E) are denoted by $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ and $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ respectively and are defined by

$$V_v : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I \text{ and } W_w : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I$$

such that

$$V_v(e)(x) = (V(e)(x), v(e)) \text{ and } W_w(e)(x) = (W(e)(x), w(e)),$$

where $V(e)(x) = \mathscr{M}(e)(x) \cup \mathscr{N}(e)(x)$, $v(e) = p(e) \cup q(e)$, $W(e)(x) = \mathscr{M}(e)(x) \cap \mathscr{N}(e)(x)$ and $w(e) = p(e) \cap q(e)$, for all $x \in X$.

Example 3. Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ be the two Type-II GPBFSS's on (X,E). By the Example 1 in $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ is defined as,

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_1) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.3,0.4,-0.2,-0.3)} \\ \frac{x_2}{(0.4,0.5,-0.6,-0.2)} \\ \frac{x_3}{(0.6,0.2,-0.1,-0.4)} \end{array}\right\}(0.5,0.4,-0.3,-0.1)\right);$$

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_2) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.8,0.7,-0.4,-0.3)} \\ \frac{x_2}{(0.6,0.4,-0.3,-0.8)} \\ \frac{x_3}{(0.5,0.3,-0.5,-0.4)} \end{array}\right\}(0.2,0.1,-0.3,-0.5)\right);$$

$$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_3) = \left(\left\{\begin{array}{c} \frac{x_1}{(0.6,0.4,-0.4,-0.1)} \\ \frac{x_2}{(0.7,0.9,-0.6,-0.4)} \\ \frac{x_3}{(0.2,0.6,-0.3,-0.2)} \end{array}\right\}(0.5,0.6,-0.3,-0.4)\right).$$

Now, $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ can be written as:

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_1) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.6,0.4,-0.3,-0.3)}\\ \frac{x_2}{(0.9,0.4,-0.7,-0.2)}\\ \frac{x_3}{(0.8,0.2,-0.2,-0.4)}\end{array}\right\}(0.6,0.4,-0.8,-0.1)\right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_2) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.8,0.4,-0.4,-0.3)}\\ \frac{x_2}{(0.6,0.4,-0.4,-0.4)}\\ \frac{x_3}{(0.5,0.3,-0.5,-0.4)}\end{array}\right\}(0.9,0.1,-0.7,-0.4)\right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_3) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.6,0.4,-0.8,-0.1)}\\ \frac{x_2}{(0.8,0.4,-0.7,-0.3)}\\ \frac{x_3}{(0.9,0.2,-0.5,-0.2)}\end{array}\right\}(0.6,0.5,-0.7,-0.3)\right).$$

Now, $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ can be written as:

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_1) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.3,0.7,-0.2,-0.8)}\\ \frac{x_2}{(0.4,0.5,-0.6,-0.5)}\\ \frac{x_3}{(0.6,0.5,-0.1,-0.9)}\end{array}\right\}(0.5,0.5,-0.3,-0.3)\right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_2) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.7,0.7,-0.2,-0.8)}\\ \frac{x_2}{(0.3,0.9,-0.3,-0.8)}\\ \frac{x_3}{(0.5,0.6,-0.2,-0.9)}\end{array}\right\}(0.2,0.2,-0.3,-0.5)\right);$$

$$\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e_3) = \left(\left\{\begin{array}{c}\frac{x_1}{(0.3,0.7,-0.4,-0.4)}\\ \frac{x_2}{(0.7,0.9,-0.6,-0.4)}\\ \frac{x_3}{(0.2,0.6,-0.3,-0.6)}\end{array}\right\}(0.5,0.6,-0.3,-0.4)\right).$$

Definition 11. A Type-II GPBFSS $\rho_\epsilon^{\mathscr{G}\mathscr{B}}(e)(x) = \left\langle \rho(e)(x), \epsilon(e)\right\rangle$ is said to a null Type-II GPBFSS $\rho_\epsilon^{\mathscr{G}\mathscr{B}} : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I$, where $\rho^+(e)(x) = (0,1)$, $\epsilon^+(e) = (0,1)$ and $\rho^-(e)(x) = (-1,0)$ and $\epsilon^-(e) = (-1,0)$, $\forall\, x \in X$.

Definition 12. A Type-II GPBFSS $\pi_\sigma^{\mathscr{G}\mathscr{B}}(e)(x) = \left\langle \pi(e)(x), \sigma(e)\right\rangle$ is said to a absolute Type-II GPBFSS $\pi_\sigma^{\mathscr{G}\mathscr{B}} : E \to P\mathscr{G}\mathscr{B}\mathscr{M}^X \times I$, where $\pi^+(e)(x) = (1,0)$, $\sigma^+(e) = (1,0)$ and $\pi^-(e)(x) = (0,-1)$ and $\sigma^-(e) = (0,-1)$, $\forall\, x \in X$.

Theorem 13. *Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ be a Type-II GPBFSS on (X,E). Then the following properties are holds:*

1. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ *and* $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
2. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \subseteq \mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ *and* $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \subseteq \mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
3. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \rho_\epsilon^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ *and* $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \rho_\epsilon^{\mathscr{G}\mathscr{B}} = \rho_\epsilon^{\mathscr{G}\mathscr{B}}$,
4. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \pi_\sigma^{\mathscr{G}\mathscr{B}} = \pi_\sigma^{\mathscr{G}\mathscr{B}}$ *and* $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \pi_\sigma^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$.

Remark 1. Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ be a Type-II GPBFSS on (X,E). If $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \neq \pi_\sigma^{\mathscr{G}\mathscr{B}}$ or $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \neq \rho_\epsilon^{\mathscr{G}\mathscr{B}}$, then $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}} \neq \pi_\sigma^{\mathscr{G}\mathscr{B}}$ and $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}} \neq \rho_\epsilon^{\mathscr{G}\mathscr{B}}$.

Theorem 14. *Let $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$, $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ and $\mathscr{O}_r^{\mathscr{G}\mathscr{B}}$ are three Type-II GPBFSS's over (X,E), then the following properties hold:*

1. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}} = \mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cup \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
2. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}} = \mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cap \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
3. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup (\mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cup \mathscr{O}_r^{\mathscr{G}\mathscr{B}}) = (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cup \mathscr{O}_r^{\mathscr{G}\mathscr{B}}$,

4. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap (\mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cap \mathscr{O}_r^{\mathscr{G}\mathscr{B}}) = (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cap \mathscr{O}_r^{\mathscr{G}\mathscr{B}}$,
5. $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}})^c = \mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}$,
6. $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}})^c = \mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}$,
7. $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cap \mathscr{M}_p^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
8. $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cup \mathscr{M}_p^{\mathscr{G}\mathscr{B}} = \mathscr{M}_p^{\mathscr{G}\mathscr{B}}$,
9. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup (\mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cap \mathscr{O}_r^{\mathscr{G}\mathscr{B}}) = (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cap (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cup \mathscr{O}_r^{\mathscr{G}\mathscr{B}})$,
10. $\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap (\mathscr{N}_q^{\mathscr{G}\mathscr{B}} \cup \mathscr{O}_r^{\mathscr{G}\mathscr{B}}) = (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) \cup (\mathscr{M}_p^{\mathscr{G}\mathscr{B}} \cap \mathscr{O}_r^{\mathscr{G}\mathscr{B}})$.

Proof. The proof follows from Definition 9 and 10. □

Definition 15. Let $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A)$ and $(\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B)$ be two Type-II GPBFSS's on (X, E). Then the operations AND is denoted by $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B)$ and is defined by

$$(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B) = (\mathscr{O}_r^{\mathscr{G}\mathscr{B}}, A \times B),$$

where $\mathscr{O}_r^{\mathscr{G}\mathscr{B}}(\gamma, \delta) = (\mathscr{O}(\gamma, \delta)(x), r(\gamma, \delta))$ such that

$$\mathscr{O}(\gamma, \delta) = \mathscr{M}(\gamma) \cap \mathscr{N}(\delta) \text{ and } r(\gamma, \delta) = p(\gamma) \cap q(\delta),$$

for all $(\gamma, \delta) \in A \times B$.

Definition 16. Let $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A)$ and $(\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B)$ be two Type-II GPBFSS's on (X, E), then the operations OR is denoted by $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B)$ and is defined by

$$(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B) = (\mathscr{O}_r^{\mathscr{G}\mathscr{B}}, A \times B),$$

where $\mathscr{O}_r^{\mathscr{G}\mathscr{B}}(\gamma, \delta) = (\mathscr{O}(\gamma, \delta)(x), r(\gamma, \delta))$ such that

$$\mathscr{O}(\gamma, \delta) = \mathscr{M}(\gamma) \cup \mathscr{N}(\delta) \text{ and } r(\gamma, \delta) = p(\gamma) \cup q(\delta),$$

for all $(\gamma, \delta) \in A \times B$.

Theorem 17. *Let $(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A)$ and $(\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B)$ be two Type-II GPBFSS's on (X, E), then*

1. $((\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B))^c = (\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B)$,
2. $((\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B))^c = (\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B)$.

Proof. 1. Suppose that

$$(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B) = (\mathscr{O}_r^{\mathscr{G}\mathscr{B}}, A \times B).$$

Now,

$$\mathscr{O}_{r^c}^{\mathscr{G}\mathscr{B}}(\gamma, \delta) = (\mathscr{O}^c(\gamma, \delta)(x), r^c(\gamma, \delta)),$$

for all $(\gamma, \delta) \in A \times B$. By Theorem 14 and Definition 15,

$$\mathscr{O}^c(\gamma, \delta) = (\mathscr{M}(\gamma) \cap \mathscr{N}(\delta))^c = \mathscr{M}^c(\gamma) \cup \mathscr{N}^c(\delta)$$

and

$$r^c(\gamma, \delta) = (p(\gamma) \cap q(\delta))^c = p^c(\gamma) \cup q^c(\delta).$$

On the other hand, given that

$$(\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B) = (\sigma_o, A \times B),$$

where $\sigma_o(\gamma, \delta) = (\sigma(\gamma, \delta)(x), o(\gamma, \delta))$ such that

$$\sigma(\gamma, \delta) = \mathscr{M}^c(\gamma) \cup \mathscr{N}^c(\delta)$$

and

$$o(\gamma,\delta) = p^c(\gamma) \cup q^c(\delta)$$

for all $(\gamma,\delta) \in A \times B$. Thus,

$$\mathscr{O}_r^c = \sigma_o.$$

Hence

$$((\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B))^c = (\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B).$$

2. Suppose that

$$(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B) = (\mathscr{O}_r^{\mathscr{G}\mathscr{B}}, A \times B).$$

Now,

$$\mathscr{O}_{r^c}^{\mathscr{G}\mathscr{B}}(\gamma,\delta) = (\mathscr{O}^c(\gamma,\delta)(x), r^c(\gamma,\delta)),$$

for all $(\gamma,\delta) \in A \times B$. By Theorem 14 and Definition 16,

$$\mathscr{O}^c(\gamma,\delta) = (\mathscr{M}(\gamma) \cup \mathscr{N}(\delta))^c = \mathscr{M}^c(\gamma) \cap \mathscr{N}^c(\delta)$$

and

$$r^c(\gamma,\delta) = (p(\gamma) \cup q(\delta))^c = p^c(\gamma) \cap q^c(\delta).$$

On the other hand, given that

$$(\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B) = (\sigma_o, A \times B),$$

where $\sigma_o(\gamma,\delta) = (\sigma(\gamma,\delta)(x), o(\gamma,\delta))$ such that

$$\sigma(\gamma,\delta) = \mathscr{M}^c(\gamma) \cap \mathscr{N}^c(\delta)$$

and

$$o(\gamma,\delta) = p^c(\gamma) \cap q^c(\delta)$$

for all $(\gamma,\delta) \in A \times B$. Thus,

$$\mathscr{O}_r^c = \sigma_o.$$

Hence

$$((\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, A) \vee (\mathscr{N}_q^{\mathscr{G}\mathscr{B}}, B))^c = (\mathscr{M}_{p^c}^{\mathscr{G}\mathscr{B}}, A) \wedge (\mathscr{N}_{q^c}^{\mathscr{G}\mathscr{B}}, B).$$

□

4. Similarity measure between two PPBFSS's

Definition 18. Let X be a non-empty set of the universe and E be a set of parameter. Suppose that $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ are two Type-II GPBFSS's on (X, E). The similarity measure between two Type-II GPBFSS's $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ and $\mathscr{N}_q^{\mathscr{G}\mathscr{B}}$ is denoted by $Sim(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, \mathscr{N}_q^{\mathscr{G}\mathscr{B}})$ and is defined as

$$Sim(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) = \left[\Delta^{\mathscr{G}\mathscr{B}}(\mathscr{M}, \mathscr{N}) \cdot \Upsilon^{\mathscr{G}\mathscr{B}}(p,q)\right]$$

such that

$$\Delta^{\mathscr{G}\mathscr{B}}(\mathscr{M}, \mathscr{N}) = \frac{\mathbb{T}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x), \mathscr{N}(e)(x)) \;+\; \mathbb{S}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x), \mathscr{N}(e)(x))}{2}$$

and

$$\Upsilon^{\mathscr{G}\mathscr{B}}(p,q) = 1 - \frac{\sum|\alpha_1 - \alpha_2|}{\sum|\alpha_1 + \alpha_2|},$$

where $\mathbb{T}^{\mathscr{GB}}(\mathscr{M}(e)(x),\mathscr{N}(e)(x))$ can be written as

$$\frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\left[\left[\zeta^{+}_{\mathscr{M}(e_i)}(x_j)\cdot\zeta^{+}_{\mathscr{N}(e_i)}(x_j)\right]+\left[\zeta^{-}_{\mathscr{M}(e_i)}(x_j)\cdot\zeta^{-}_{\mathscr{N}(e_i)}(x_j)\right]\right]}{\sum_{i=1}^{n}\left[\left[1-\sqrt{\left[(1-\zeta^{2+}_{\mathscr{M}(e_i)}(x_j))\cdot(1-\zeta^{2+}_{\mathscr{N}(e_i)}(x_j))\right]}\right]+\left[1-\sqrt{\left[(1-\zeta^{2-}_{\mathscr{M}(e_i)}(x_j))\cdot(1-\zeta^{2-}_{\mathscr{N}(e_i)}(x_j))\right]}\right]\right]}$$

and $\mathbb{S}^{\mathscr{GB}}(\mathscr{M}(e)(x),\mathscr{N}(e)(x))$ can be written as

$$\frac{1}{m}\sum_{j=1}^{m}\sqrt{1-\frac{\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{M}(e_i)}(x_j)-\xi^{2+}_{\mathscr{N}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{M}(e_i)}(x_j)-\xi^{2-}_{\mathscr{N}(e_i)}(x_j)\right|\right]}{\sum_{i=1}^{n}\left[\left[1+\left[(\xi^{2+}_{\mathscr{M}(e_i)}(x_j))\cdot(\xi^{2+}_{\mathscr{N}(e_i)}(x_j))\right]\right]+\left[1+\left[(\xi^{2-}_{\mathscr{M}(e_i)}(x_j))\cdot(\xi^{2-}_{\mathscr{N}(e_i)}(x_j))\right]\right]\right]}},$$

$$\alpha_1=\frac{\zeta^{2+}_{p(e_i)}+\zeta^{2-}_{p(e_i)}}{\left[\zeta^{2+}_{p(e_i)}+\xi^{2+}_{p(e_i)}\right]+\left[\zeta^{2-}_{p(e_i)}+\xi^{2-}_{p(e_i)}\right]}$$

and

$$\alpha_2=\frac{\zeta^{2+}_{q(e_i)}+\zeta^{2-}_{q(e_i)}}{\left[\zeta^{2+}_{q(e_i)}+\xi^{2+}_{q(e_i)}\right]+\left[\zeta^{2-}_{q(e_i)}+\xi^{2-}_{q(e_i)}\right]}.$$

Theorem 19. *Let $\mathscr{M}_p^{\mathscr{GB}}$, $\mathscr{N}_q^{\mathscr{GB}}$ and $\mathscr{O}_r^{\mathscr{GB}}$ be the any three Type-II GPBFSS's over (X,E). Then the following statements hold:*

1. $Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{N}_q^{\mathscr{GB}})=Sim(\mathscr{N}_q^{\mathscr{GB}},\mathscr{M}_p^{\mathscr{GB}})$,
2. $0\le Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{N}_q^{\mathscr{GB}})\le 1$,
3. $\mathscr{M}_p^{\mathscr{GB}}=\mathscr{N}_q^{\mathscr{GB}}\implies Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{N}_q^{\mathscr{GB}})=1$,
4. $\mathscr{M}_p^{\mathscr{GB}}\subseteq\mathscr{N}_q^{\mathscr{GB}}\subseteq\mathscr{O}_r^{\mathscr{GB}}\implies Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{O}_r^{\mathscr{GB}})\le Sim(\mathscr{N}_q^{\mathscr{GB}},\mathscr{O}_r^{\mathscr{GB}})$,
5. $\mathscr{M}_p^{\mathscr{GB}}\cap\mathscr{N}_q^{\mathscr{GB}}=\{\phi\}\Leftrightarrow Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{N}_q^{\mathscr{GB}})=0$.

Proof. The statements (1), (2) and (5) are trivial.
(3) Given that $\mathscr{M}_p^{\mathscr{GB}}=\mathscr{N}_q^{\mathscr{GB}}$. Now,

$$\mathbb{T}^{\mathscr{GB}}(\mathscr{M}(e)(x_j),\mathscr{N}(e)(x_j))=\frac{\sum_{i=1}^{n}\left[\zeta^{2+}_{\mathscr{M}(e_i)}(x_j)+\zeta^{2-}_{\mathscr{M}(e_i)}(x_j)\right]}{\sum_{i=1}^{n}\left[\zeta^{2+}_{\mathscr{M}(e_i)}(x_j)+\zeta^{2-}_{\mathscr{M}(e_i)}(x_j)\right]}=1.$$

Hence, $\mathbb{T}^{\mathscr{GB}}(\mathscr{M}(e)(x),\mathscr{N}(e)(x))=\frac{1}{m}[1+1+...+1(\text{m times})]=1$. Now, $\mathbb{S}^{\mathscr{GB}}(\mathscr{M}(e)(x_j),\mathscr{N}(e)(x_j))=\sqrt{(1-0)}=1$. Hence, $\mathbb{S}^{\mathscr{GB}}(\mathscr{M}(e)(x),\mathscr{N}(e)(x))=\frac{1}{m}[1+1+...+1(\text{m times})]=1$. Thus, $\Delta^{\mathscr{GB}}(\mathscr{M},\mathscr{N})=\frac{1+1}{2}=1$ and $\Upsilon^{\mathscr{GB}}(p,q)=1$. Hence $Sim(\mathscr{M}_p^{\mathscr{GB}},\mathscr{N}_q^{\mathscr{GB}})=1$. This proves (3).
(4) Given that

$$\mathscr{M}_p^{\mathscr{GB}}\subseteq\mathscr{N}_q^{\mathscr{GB}}\implies\left\{\begin{array}{l}\zeta^{+}_{\mathscr{M}(e)}(x_j)\le\zeta^{+}_{\mathscr{N}(e)}(x_j),\ \xi^{+}_{\mathscr{M}(e)}(x_j)\ge\xi^{+}_{\mathscr{N}(e)}(x_j),\\ \zeta^{-}_{\mathscr{M}(e)}(x_j)\ge\zeta^{-}_{\mathscr{N}(e)}(x_j),\ \xi^{-}_{\mathscr{M}(e)}(x_j)\le\xi^{-}_{\mathscr{N}(e)}(x_j),\\ \zeta^{+}_{p(e)}(x_j)\le\zeta^{+}_{q(e)}(x_j),\ \xi^{+}_{p(e)}(x_j)\ge\xi^{+}_{q(e)}(x_j),\\ \zeta^{-}_{p(e)}(x_j)\ge\zeta^{-}_{q(e)}(x_j),\ \xi^{-}_{p(e)}(x_j)\le\xi^{-}_{q(e)}(x_j),\end{array}\right\}$$

$$\mathscr{M}_p^{\mathscr{GB}}\subseteq\mathscr{O}_r^{\mathscr{GB}}\implies\left\{\begin{array}{l}\zeta^{+}_{\mathscr{M}(e)}(x_j)\le\zeta^{+}_{\mathscr{O}(e)}(x_j),\ \xi^{+}_{\mathscr{M}(e)}(x_j)\ge\xi^{+}_{\mathscr{O}(e)}(x_j),\\ \zeta^{-}_{\mathscr{M}(e)}(x_j)\ge\zeta^{-}_{\mathscr{O}(e)}(x_j),\ \xi^{-}_{\mathscr{M}(e)}(x_j)\le\xi^{-}_{\mathscr{O}(e)}(x_j),\\ \zeta^{+}_{p(e)}(x_j)\le\zeta^{+}_{r(e)}(x_j),\ \xi^{+}_{p(e)}(x_j)\ge\xi^{+}_{r(e)}(x_j),\\ \zeta^{-}_{p(e)}(x_j)\ge\zeta^{-}_{r(e)}(x_j),\ \xi^{-}_{p(e)}(x_j)\le\xi^{-}_{r(e)}(x_j),\end{array}\right\}$$

$$\mathcal{N}_q^{\mathcal{GB}} \subseteq \mathcal{O}_r^{\mathcal{GB}} \implies \begin{Bmatrix} \zeta^+_{\mathcal{N}(e)}(x_j) \leq \zeta^+_{\mathcal{O}(e)}(x_j),\ \xi^+_{\mathcal{N}(e)}(x_j) \geq \xi^+_{\mathcal{O}(e)}(x_j), \\ \zeta^-_{\mathcal{N}(e)}(x_j) \geq \zeta^-_{\mathcal{O}(e)}(x_j),\ \xi^-_{\mathcal{N}(e)}(x_j) \leq \xi^-_{\mathcal{O}(e)}(x_j), \\ \zeta^+_{q(e)}(x_j) \leq \zeta^+_{r(e)}(x_j),\ \xi^+_{q(e)}(x_j) \geq \xi^+_{r(e)}(x_j), \\ \zeta^-_{q(e)}(x_j) \geq \zeta^-_{r(e)}(x_j),\ \xi^-_{q(e)}(x_j) \leq \xi^-_{r(e)}(x_j), \end{Bmatrix}$$

Clearly, $\zeta^+_{\mathcal{M}(e)}(x_j)\cdot\zeta^+_{\mathcal{O}(e)}(x_j) \leq \zeta^+_{\mathcal{N}(e)}(x_j)\cdot\zeta^+_{\mathcal{O}(e)}(x_j)$ and $\zeta^-_{\mathcal{M}(e)}(x_j)\cdot\zeta^-_{\mathcal{O}(e)}(x_j) \leq \zeta^-_{\mathcal{N}(e)}(x_j)\cdot\zeta^-_{\mathcal{O}(e)}(x_j)$ implies that

$$\sum_{i=1}^{n}\Big[\big[\zeta^+_{\mathcal{M}(e_i)}(x_j)\cdot\zeta^+_{\mathcal{O}(e_i)}(x_j)\big]+\big[\zeta^-_{\mathcal{M}(e_i)}(x_j)\cdot\zeta^-_{\mathcal{O}(e_i)}(x_j)\big]\Big] \leq \sum_{i=1}^{n}\Big[\big[\zeta^+_{\mathcal{N}(e_i)}(x_j)\cdot\zeta^+_{\mathcal{O}(e_i)}(x_j)\big]+\big[\zeta^-_{\mathcal{N}(e_i)}(x_j)\cdot\zeta^-_{\mathcal{O}(e_i)}(x_j)\big]\Big]. \tag{1}$$

Clearly, $(\zeta^+_{\mathcal{M}(e)}(x_j))^2 \leq (\zeta^+_{\mathcal{N}(e)}(x_j))^2$ and $(\zeta^-_{\mathcal{M}(e)}(x_j))^2 \leq (\zeta^-_{\mathcal{N}(e)}(x_j))^2$ implies that

$$\Big[(1-(\zeta^+_{\mathcal{M}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\Big] \geq \Big[(1-(\zeta^+_{\mathcal{N}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\Big]$$

and

$$\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{M}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big] \leq \Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{N}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big]. \tag{2}$$

Similarly,

$$\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{M}(e)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big] \leq \Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{N}(e)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big] \tag{3}$$

Adding Eqs (2) and (3) we get,

$$\begin{aligned}&\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{M}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{M}(e)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big]\\ &\leq \Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{N}(e)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{N}(e)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e)}(x_j))^2)\big]}\,\Big].\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{n}\Bigg[\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{M}(e_i)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{M}(e_i)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]\Bigg]\\ &\leq \sum_{i=1}^{n}\Bigg[\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{N}(e_i)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{N}(e_i)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]\Bigg]\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{m}\sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n}\Bigg[\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{M}(e_i)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{M}(e_i)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]\Bigg]\Bigg]\\ &\leq \frac{1}{m}\sum_{j=1}^{m}\Bigg[\sum_{i=1}^{n}\Bigg[\Big[1-\sqrt{\big[(1-(\zeta^+_{\mathcal{N}(e_i)}(x_j))^2)\cdot(1-(\zeta^+_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]+\Big[1-\sqrt{\big[(1-(\zeta^-_{\mathcal{N}(e_i)}(x_j))^2)\cdot(1-(\zeta^-_{\mathcal{O}(e_i)}(x_j))^2)\big]}\,\Big]\Bigg]\Bigg].\end{aligned} \tag{4}$$

Dividing Eq. (1) by Eq. (4),

$$\frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\Big[\big[\zeta^+_{\mathcal{M}(e_i)}(x_j)\ \cdot\ \zeta^+_{\mathcal{O}(e_i)}(x_j)\big]+\big[\zeta^-_{\mathcal{M}(e_i)}(x_j)\ \cdot\ \zeta^-_{\mathcal{O}(e_i)}(x_j)\big]\Big]}{\sum_{i=1}^{n}\Bigg[\Big[1-\sqrt{\big[(1-\zeta^{2+}_{\mathcal{M}(e_i)}(x_j))\ \cdot\ (1-\zeta^{2+}_{\mathcal{O}(e_i)}(x_j))\big]}\,\Big]+\Big[1-\sqrt{\big[(1-\zeta^{2-}_{\mathcal{M}(e_i)}(x_j))\ \cdot\ (1-\zeta^{2-}_{\mathcal{O}(e_i)}(x_j))\big]}\,\Big]\Bigg]}$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\frac{\sum_{i=1}^{n}\left[\left[\zeta^{+}_{\mathscr{N}(e_i)}(x_j)\ \cdot\ \zeta^{+}_{\mathscr{O}(e_i)}(x_j)\right]+\left[\zeta^{-}_{\mathscr{N}(e_i)}(x_j)\ \cdot\ \zeta^{-}_{\mathscr{O}(e_i)}(x_j)\right]\right]}{\sum_{i=1}^{n}\left[\left[1-\sqrt{\left[(1-\zeta^{2+}_{\mathscr{N}(e_i)}(x_j))\ \cdot\ (1-\zeta^{2+}_{\mathscr{O}(e_i)}(x_j))\right]}\right]+\left[1-\sqrt{\left[(1-\zeta^{2-}_{\mathscr{N}(e_i)}(x_j))\ \cdot\ (1-\zeta^{2-}_{\mathscr{O}(e_i)}(x_j))\right]}\right]\right]}. \quad (5)$$

Clearly, $\xi^{2+}_{\mathscr{M}(e)}(x_j) \geq \xi^{2+}_{\mathscr{N}(e)}(x_j) \geq \xi^{2+}_{\mathscr{O}(e)}(x_j)$ and $\xi^{2-}_{\mathscr{M}(e)}(x_j) \geq \xi^{2-}_{\mathscr{N}(e)}(x_j) \geq \xi^{2-}_{\mathscr{O}(e)}(x_j)$. Thus,

$$\left[\xi^{2+}_{\mathscr{M}(e)}(x_j)-\xi^{2+}_{\mathscr{O}(e)}(x_j)\right] \geq \left[\xi^{2+}_{\mathscr{N}(e)}(x_j)-\xi^{2+}_{\mathscr{O}(e)}(x_j)\right]$$

and

$$\left[\xi^{2-}_{\mathscr{M}(e)}(x_j)-\xi^{2-}_{\mathscr{O}(e)}(x_j)\right] \geq \left[\xi^{2-}_{\mathscr{N}(e)}(x_j)-\xi^{2-}_{\mathscr{O}(e)}(x_j)\right].$$

Hence

$$\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{M}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{M}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right]$$
$$\geq \sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{N}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{N}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right] \quad (6)$$

Also,

$$\left[\xi^{2+}_{\mathscr{M}(e)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e)}(x_j)\right] \geq \left[\xi^{2+}_{\mathscr{N}(e)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e)}(x_j)\right]$$

and

$$\left[\xi^{2-}_{\mathscr{M}(e)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e)}(x_j)\right] \geq \left[\xi^{2-}_{\mathscr{N}(e)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e)}(x_j)\right].$$

Hence

$$\sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]$$
$$\geq \sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]. \quad (7)$$

Dividing Eq. (6) by Eq. (7), we get

$$\frac{\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{M}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{M}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right]}{\sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]}$$
$$\geq \frac{\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{N}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{N}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right]}{\sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]}$$

and

$$\frac{1}{m}\sum_{j=1}^{m}\sqrt{1-\frac{\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{M}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{M}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right]}{\sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{M}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]}}$$

$$\leq \frac{1}{m}\sum_{j=1}^{m}\sqrt{1-\frac{\sum_{i=1}^{n}\left[\left|\xi^{2+}_{\mathscr{N}(e_i)}(x_j)-\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right|+\left|\xi^{2-}_{\mathscr{N}(e_i)}(x_j)-\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right|\right]}{\sum_{i=1}^{n}\left[\left[1+\left[\xi^{2+}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2+}_{\mathscr{O}(e_i)}(x_j)\right]\right]+\left[1+\left[\xi^{2-}_{\mathscr{N}(e_i)}(x_j)\cdot\xi^{2-}_{\mathscr{O}(e_i)}(x_j)\right]\right]\right]}}. \tag{8}$$

Adding Eqs (5) and (8) and divided by 2,

$$\Delta^{\mathscr{G}\mathscr{B}}(\mathscr{M},\mathscr{O}) \leq \Delta^{\mathscr{G}\mathscr{B}}(\mathscr{N},\mathscr{O}). \tag{9}$$

Clearly $\alpha_1 \leq \alpha_2 \leq \alpha_3$, where

$$\alpha_1 = \frac{\zeta^{2+}_{p(e_i)}+\zeta^{2-}_{p(e_i)}}{\left[\zeta^{2+}_{p(e_i)} + \xi^{2+}_{p(e_i)}\right]+\left[\zeta^{2-}_{p(e_i)} + \xi^{2-}_{p(e_i)}\right]},$$
$$\alpha_2 = \frac{\zeta^{2+}_{q(e_i)}+\zeta^{2-}_{q(e_i)}}{\left[\zeta^{2+}_{q(e_i)} + \xi^{2+}_{q(e_i)}\right]+\left[\zeta^{2-}_{q(e_i)} + \xi^{2-}_{q(e_i)}\right]},$$
$$\alpha_3 = \frac{\zeta^{2+}_{r(e_i)}+\zeta^{2-}_{r(e_i)}}{\left[\zeta^{2+}_{r(e_i)} + \xi^{2+}_{r(e_i)}\right]+\left[\zeta^{2-}_{r(e_i)} + \xi^{2-}_{r(e_i)}\right]}.$$

Clearly,

$$\alpha_1 - \alpha_3 \leq \alpha_2 - \alpha_3.$$

Thus,

$$\left|\alpha_2-\alpha_3\right| \leq \left|\alpha_1-\alpha_3\right| \implies -\sum_{i=1}^{n}\left|\alpha_1-\alpha_3\right| \leq -\sum_{i=1}^{n}\left|\alpha_2-\alpha_3\right|. \tag{10}$$

Since

$$\sum_{i=1}^{n}\left|\alpha_1+\alpha_3\right| \leq \sum_{i=1}^{n}\left|\alpha_2+\alpha_3\right| \tag{11}$$

Dividing Eq. (10) by Eq. (11), we get

$$\frac{-\sum_{i=1}^{n}\left|\alpha_1-\alpha_3\right|}{\sum_{i=1}^{n}\left|\alpha_1+\alpha_3\right|} \leq \frac{-\sum_{i=1}^{n}\left|\alpha_2-\alpha_3\right|}{\sum_{i=1}^{n}\left|\alpha_2+\alpha_3\right|} \implies 1-\frac{\sum_{i=1}^{n}\left|\alpha_1-\alpha_3\right|}{\sum_{i=1}^{n}\left|\alpha_1+\alpha_3\right|} \leq 1-\frac{\sum_{i=1}^{n}\left|\alpha_2-\alpha_3\right|}{\sum_{i=1}^{n}\left|\alpha_2+\alpha_3\right|}.$$

Hence

$$\Upsilon^{\mathscr{G}\mathscr{B}}(p,r) \leq \Upsilon^{\mathscr{G}\mathscr{B}}(q,r) \tag{12}$$

Multiplying Eqs (9) and (12),

$$\Delta^{\mathscr{G}\mathscr{B}}(\mathscr{M},\mathscr{O})\cdot\Upsilon^{\mathscr{G}\mathscr{B}}(p,r) \leq \Delta^{\mathscr{G}\mathscr{B}}(\mathscr{N},\mathscr{O})\cdot\Upsilon^{\mathscr{G}\mathscr{B}}(q,r).$$

Hence

$$Sim(\mathscr{M}^{\mathscr{G}\mathscr{B}}_p,\mathscr{O}^{\mathscr{G}\mathscr{B}}_r) \leq Sim(\mathscr{N}^{\mathscr{G}\mathscr{B}}_q,\mathscr{O}^{\mathscr{G}\mathscr{B}}_r).$$

This proves (4). □

Example 4. Calculate the similarity measure between the two Type-II GPBFSS's namely $\mathscr{M}^{\mathscr{G}\mathscr{B}}_p$ and $\mathscr{N}^{\mathscr{G}\mathscr{B}}_q$. Let $X = \{x_1, x_2, x_3\}$ and $E = \{e_1, e_2, e_3\}$ can be defined as below:

$\mathscr{M}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{M}(e)(x_1)$	$(0.6, 0.65, -0.3, -0.8)$	$(0.9, 0.35, -0.7, -0.5)$	$(0.8, 0.45, -0.2, -0.9)$
$\mathscr{M}(e)(x_2)$	$(0.55, 0.55, -0.35, -0.75)$	$(0.85, 0.25, -0.75, -0.45)$	$(0.75, 0.35, -0.25, -0.85)$
$\mathscr{M}(e)(x_3)$	$(0.45, 0.55, -0.55, -0.7)$	$(0.75, 0.25, -0.45, -0.65)$	$(0.65, 0.35, -0.15, -0.75)$
$p(e)$	$(0.6, 0.5, -0.8, -0.3)$	$(0.8, 0.3, -0.6, -0.5)$	$(0.7, 0.4, -0.8, -0.6)$

$\mathscr{N}_q^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{N}(e)(x_1)$	$(0.3, 0.35, -0.2, -0.3)$	$(0.4, 0.45, -0.6, -0.2)$	$(0.6, 0.25, -0.1, -0.4)$
$\mathscr{N}(e)(x_2)$	$(0.25, 0.3, -0.25, -0.2)$	$(0.35, 0.4, -0.65, -0.15)$	$(0.55, 0.2, -0.15, -0.35)$
$\mathscr{N}(e)(x_3)$	$(0.35, 0.4, -0.2, -0.45)$	$(0.55, 0.45, -0.35, -0.25)$	$(0.6, 0.4, -0.25, -0.55)$
$q(e)$	$(0.5, 0.35, -0.3, -0.1)$	$(0.6, 0.15, -0.4, -0.2)$	$(0.4, 0.25, -0.5, -0.6)$

Now,

$$\mathbb{T}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_1), \mathscr{N}(e)(x_1)) = \frac{1.52}{1.876484} = 0.810025,$$

$$\mathbb{T}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_2), \mathscr{N}(e)(x_2)) = \frac{1.46}{1.778538} = 0.820899$$

and

$$\mathbb{T}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_3), \mathscr{N}(e)(x_3)) = \frac{1.265}{1.390973} = 0.909436.$$

Hence,

$$\mathbb{T}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x), \mathscr{N}(e)(x)) = \frac{0.810025 + 0.820899 + 0.909436}{3} = 0.846787.$$

Now,

$$\mathbb{S}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_1), \mathscr{N}(e)(x_1)) = \sqrt{1 - \frac{1.93}{6.286419}} = 0.83246,$$

$$\mathbb{S}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_2), \mathscr{N}(e)(x_2)) = \sqrt{1 - \frac{1.695}{6.157688}} = 0.851313$$

and

$$\mathbb{S}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x_3), \mathscr{N}(e)(x_3)) = \sqrt{1 - \frac{1.2275}{6.376444}} = 0.898607.$$

Hence

$$\mathbb{S}^{\mathscr{G}\mathscr{B}}(\mathscr{M}(e)(x), \mathscr{N}(e)(x)) = \frac{0.83246 + 0.851313 + 0.898607}{3} = 0.860793.$$

Thus,

$$\Delta^{\mathscr{G}\mathscr{B}}(\mathscr{M}, \mathscr{N}) = \frac{0.846787 + 0.860793}{2} = 0.85379$$

and

$$\Upsilon^{\mathscr{G}\mathscr{B}}(p, q) = 1 - \frac{0.365483}{4.282159} = 0.91465.$$

Hence,

$$Sim(\mathscr{M}_p^{\mathscr{G}\mathscr{B}}, \mathscr{N}_q^{\mathscr{G}\mathscr{B}}) = 0.85379 \times 0.91465 = 0.780919.$$

5. Application for medical diagnosis

Decision making problems are a big part of human society and applied widely to practical fields like education, economics, management, engineering and Hospital. However, with the development of science and technology, the uncertainty also plays a dominant role at some point of the decision making analysis. In this application, we present a method for a medical diagnosis problem based on the proposed similarity measure of Type-II GPBFSS's. This technique of similarity measure between two Type-II GPBFSS's can be applied to detect whether an ill person is suffering from a certain disease or not. We first give the following remark;

Remark 2. Let $\mathscr{M}_p$ and $\mathscr{N}_q$ be two Type-II GPBFSS's over the same soft universe (X, E). We call the two Type-II GPBFSS 's to be significantly similar if $Sim(\mathscr{M}_p, \mathscr{N}_q) \geq 70$.

We first construct a Type-II GPBFSS for the illness with the help of a medical person and a Type-II GPBFSS for the ill person. Then, we calculate the similarity measure between two Type-II GPBFSS's. If they are significantly similar, then we infer that the person may have disease, and otherwise not.

5.1. Algorithm

The algorithm for the selection of the best choice is given as:

1. Input the Type-II GPBFSS $\mathscr{M}_p^{\mathscr{G}\mathscr{B}}$ in tabular form.
2. Input the set of choice parameters $A \subseteq E$.
3. Compute the values of $\mathbb{T}^{\mathscr{G}\mathscr{B}}$ and $\mathbb{S}^{\mathscr{G}\mathscr{B}}$.
4. Calculate the $\Delta^{\mathscr{G}\mathscr{B}}$ value by taking $\frac{\mathbb{T}^{\mathscr{G}\mathscr{B}}+\mathbb{S}^{\mathscr{G}\mathscr{B}}}{2}$.
5. Determine the value $\Upsilon^{\mathscr{G}\mathscr{B}} = 1 - \frac{\sum|\alpha_1-\alpha_2|}{\sum|\alpha_1+\alpha_2|}$.
6. Compute similarity measure $\Delta^{\mathscr{G}\mathscr{B}} \cdot \Upsilon^{\mathscr{G}\mathscr{B}}$.
7. Select similarity measure using suitable criteria for significantly similar.
8. Finally, decision is to choose as the solution to the problem.
9. End.

5.2. Case study

Suppose that there are five patients $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3, \mathscr{P}_4$ and $\mathscr{P}_5$ in a hospital with certain symptoms of Scrub Typhus. Let the universal set contain only three elements. That is $X = \{x_1 :$ severe, x_2: mild, $x_3 :$ no$\}$ and the set of parameters E is the set of certain symptoms of Scrub Typhus is represented by $E = \{e_1 :$ Fever and chills, e_2: headache, $e_3 :$ muscle pain, $e_4 :$ mental changes, e_5: enlarged lymph nodes$\}$. Table 1 represents the Scrub Typhus prepared with the help of a medical person.

Table 1. Type-II GPBFSS for pneumonia(Scrub Typhus)

$\mathscr{L}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{L}(e)(x_1)$	$(0.92, 0.18, -0.91, -0.25)$	$(0.83, 0.25, -0.72, -0.35)$	$(0.91, 0.35, -0.85, -0.25)$
$\mathscr{L}(e)(x_2)$	$(0.91, 0.15, -0.9, -0.2)$	$(0.82, 0.23, -0.7, -0.3)$	$(0.9, 0.33, -0.81, -0.22)$
$\mathscr{L}(e)(x_3)$	$(0.85, 0.25, -0.8, -0.25)$	$(0.8, 0.3, -0.75, -0.35)$	$(0.7, 0.4, -0.8, -0.2)$
$l(e)$	$(1, 0, -1, 0)$	$(1, 0, -1, 0)$	$(1, 0, -1, 0)$
	e_4	e_5	
	$(0.84, 0.15, -0.92, -0.35)$	$(0.93, 0.25, -0.73, -0.36)$	
	$(0.8, 0.13, -0.9, -0.31)$	$(0.9, 0.21, -0.71, -0.34)$	
	$(0.75, 0.35, -0.85, -0.3)$	$(0.85, 0.25, -0.75, -0.4)$	
	$(1, 0, -1, 0)$	$(1, 0, -1, 0)$	

We construct the Type-II GPBFSS's for five patients under consideration as in Tables 2-6.

Table 2. Type-II GPBFSS for the ill person $\mathscr{P}_1$

$\mathscr{A}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{A}(e)(x_1)$	$(0.75, 0.45, -0.5, -0.85)$	$(0.45, 0.75, -0.5, -0.75)$	$(0.75, 0.65, -0.6, -0.45)$
$\mathscr{A}(e)(x_2)$	$(0.6, 0.4, -0.7 - 0.35)$	$(0.5, 0.3, 0.4 - 0.65)$	$(0.7, 0.5, -0.45 - 0.55)$
$\mathscr{A}(e)(x_3)$	$(0.75, 0.3, -0.55, -0.3)$	$(0.55, 0.7, -0.45, -0.5)$	$(0.45, 0.5, -0.7, -0.3)$
$p_1(e)$	$(0.8, 0.15, -0.55, -0.45)$	$(0.7, 0.25, -0.65, -0.55)$	$(0.6, 0.5, -0.4, -0.6)$
	e_4	e_5	
	$(0.35, 0.65, -0.7, -0.55)$	$(0.65, 0.55, -0.3, -0.65)$	
	$(0.6, 0.55, -0.6 - 0.45)$	$(0.8, 0.45, -0.5 - 0.65)$	
	$(0.55, 0.6, -0.6, -0.4)$	$(0.65, 0.5, -0.45, -0.5)$	
	$(0.4, 0.6, -0.6, -0.3)$	$(0.5, 0.7, -0.5, -0.4)$	

Table 3. Type-II GPBFSS for the ill person$\mathscr{P}_2$

$\mathscr{B}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{B}(e)(x_1)$	$(0.6, 0.45, -0.7, -0.65)$	$(0.55, 0.35, -0.65, -0.55)$	$(0.7, 0.4, -0.75, -0.45)$
$\mathscr{B}(e)(x_2)$	$(0.62, 0.4, -0.7, -0.35)$	$(0.7, 0.65, -0.45, -0.5)$	$(0.65, 0.5, -0.5, -0.55)$
$\mathscr{B}(e)(x_3)$	$(0.75, 0.35, -0.65, -0.45)$	$(0.5, 0.6, -0.55, -0.63)$	$(0.55, 0.65, -0.84, -0.4)$
$p_2(e)$	$(0.8, 0.15, -0.55, -0.5)$	$(0.7, 0.35, -0.7, -0.45)$	$(0.65, 0.65, -0.45, -0.75)$
	e_4	e_5	
	$(0.5, 0.65, -0.6, -0.5)$	$(0.75, 0.45, -0.7, -0.65)$	
	$(0.7, 0.45, -0.6, -0.45)$	$(0.8, 0.35, -0.55, -0.6)$	
hline	$(0.65, 0.45, -0.73, -0.55)$ $(0.45, 0.75, -0.7, -0.35)$	$(0.8, 0.55, -0.52, -0.6)$ $(0.3, 0.85, -0.45, -0.65)$	

Table 4. Type-II GPBFSS for the ill person $\mathscr{P}_3$

$\mathscr{C}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{C}(e)(x_1)$	$(0.8, 0.45, -0.75, -0.6)$	$(0.7, 0.35, -0.7, -0.65)$	$(0.75, 0.25, -0.8, -0.6)$
$\mathscr{C}(e)(x_2)$	$(0.8, 0.55, -0.75, -0.4)$	$(0.75, 0.6, -0.5, -0.65)$	$(0.7, 0.6, -0.65, -0.5)$
$\mathscr{C}(e)(x_3)$	$(0.82, 0.25, -0.65, -0.45)$	$(0.7, 0.55, -0.55, -0.65)$	$(0.65, 0.4, -0.8, -0.4)$
$p_3(e)$	$(0.8, 0.25, -0.65, -0.45)$	$(0.7, 0.25, -0.7, -0.4)$	$(0.6, 0.65, -0.6, -0.55)$
	e_4	e_5	
	$(0.8, 0.3, -0.75, -0.55)$	$(0.6, 0.4, -0.6, -0.75)$	
	$(0.65, 0.7, -0.7, -0.4)$	$(0.85, 0.55, -0.6, -0.55)$	
	$(0.6, 0.3, -0.75, -0.6)$	$(0.8, 0.35, -0.5, -0.65)$	
	$(0.5, 0.45, -0.85, -0.35)$	$(0.4, 0.5, -0.65, -0.5)$	

Table 5. Type-II GPBFSS for the ill person $\mathscr{P}_4$

$\mathscr{D}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{D}(e)(x_1)$	$(0.7, 0.45, -0.6, -0.75)$	$(0.5, 0.75, -0.65, -0.75)$	$(0.6, 0.6, -0.7, -0.4)$
$\mathscr{D}(e)(x_2)$	$(0.6, 0.7, -0.8, -0.4)$	$(0.55, 0.6, -0.5, -0.7)$	$(0.7, 0.65, -0.5, -0.6)$
$\mathscr{D}(e)(x_3)$	$(0.8, 0.45, -0.6, -0.4)$	$(0.65, 0.8, -0.5, -0.6)$	$(0.55, 0.6, -0.85, -0.4)$
$p_4(e)$	$(0.85, 0.2, -0.65, -0.5)$	$(0.8, 0.3, -0.75, -0.6)$	$(0.75, 0.6, -0.5, -0.85)$
	e_4	e_5	
	$(0.75, 0.55, -0.65, -0.5)$	$(0.7, 0.35, -0.4, -0.6)$	
	$(0.6, 0.7, -0.8, -0.5)$	$(0.8, 0.55, -0.6, -0.7)$	
	$(0.6, 0.7, -0.7, -0.55)$	$(0.7, 0.6, -0.5, -0.6)$	
	$(0.5, 0.75, -0.8, -0.45)$	$(0.55, 0.8, -0.65, -0.4)$	

Table 6. Type-II GPBFSS for the ill person $\mathscr{P}_5$

$\mathscr{E}_p^{\mathscr{G}\mathscr{B}}(e)$	e_1	e_2	e_3
$\mathscr{E}(e)(x_1)$	$(0.85, 0.4, -0.6, -0.8)$	$(0.5, 0.7, -0.55, -0.7)$	$(0.8, 0.65, -0.75, -0.4)$
$\mathscr{E}(e)(x_2)$	$(0.75, 0.6, -0.8, -0.45)$	$(0.5, 0.65, -0.5, -0.7)$	$(0.75, 0.4, -0.65, -0.6)$
$\mathscr{E}(e)(x_3)$	$(0.65, 0.45, -0.6, -0.4)$	$(0.55, 0.7, -0.55, -0.6)$	$(0.45, 0.6, -0.8, -0.45)$
$p_5(e)$	$(0.9, 0.2, -0.6, -0.5)$	$(0.75, 0.45, -0.7, -0.45)$	$(0.7, 0.65, -0.55, -0.8)$
	e_4	e_5	
	$(0.45, 0.6, -0.8, -0.5)$	$(0.7, 0.55, -0.4, -0.65)$	
	$(0.55, 0.75, -0.8, -0.5)$	$(0.6, 0.5, -0.45, -0.75)$	
	$(0.6, 0.7, -0.7, -0.55)$	$(0.75, 0.65, -0.5, -0.65)$	
	$(0.45, 0.55, -0.8, -0.4)$	$(0.65, 0.35, -0.6, -0.75)$	

The generalized bipolar fuzzy values in Tables 2-6 are provided by the experts,depending on their assessment of the alternatives against the criteria under consideration. We calculate the similarity measure of Type-II GPBFSS's in Tables 2-6 with the one in Table 1. Calculating the similarity measure for $\mathscr{P}_1$ to $\mathscr{P}_5$ ill persons are given below the Table 7.

Table 7. Similarity measure for $\mathscr{P}_1$ to $\mathscr{P}_5$ ill persons

	$\mathbb{T}^{\mathscr{G}\mathscr{B}}(x_1)$	$\mathbb{S}^{\mathscr{G}\mathscr{B}}(x_1)$	$\mathbb{T}^{\mathscr{G}\mathscr{B}}(x_2)$	$\mathbb{S}^{\mathscr{G}\mathscr{B}}(x_2)$	$\mathbb{T}^{\mathscr{G}\mathscr{B}}(x_3)$	$\mathbb{S}^{\mathscr{G}\mathscr{B}}(x_3)$
$(\mathscr{L},\mathscr{A})$	0.823604	0.823796	0.774984	0.904968	0.903511	0.934002
$(\mathscr{L},\mathscr{B})$	0.89267	0.90181	0.903051	0.909339	0.953667	0.905286
$(\mathscr{L},\mathscr{C})$	0.945782	0.899062	0.95394	0.870195	0.970414	0.929305
$(\mathscr{L},\mathscr{D})$	0.879251	0.862084	0.919538	0.830471	0.94947	0.875255
$(\mathscr{L},\mathscr{E})$	0.897963	0.84803	0.9162	0.841527	0.934313	0.874316
	$\mathbb{T}^{\mathscr{G}\mathscr{B}}$	$\mathbb{S}^{\mathscr{G}\mathscr{B}}$	$\Delta^{\mathscr{G}\mathscr{B}}$	$\Upsilon^{\mathscr{G}\mathscr{B}}$	Similarity	
	0.834033	0.887589	0.860811	0.742551	0.639196	
	0.916463	0.905479	0.910971	0.68785	0.626611	
	0.956712	0.899521	0.928116	0.8097	0.751495	
	0.916086	0.855937	0.870668	0.744906	0.648565	
	0.916159	0.854624	0.885392	0.769917	0.681678	

We find that the similarity measure of the first two patients and last two patients are < 0.70, but the similarity measure of third patient $\mathscr{P}_3$ is $(\mathscr{L},\mathscr{P}_3) = 0.751495 \geq 0.70$. Hence these two Type-II GPBFSS's are significantly similar. Therefore, we conclude that the patient $\mathscr{P}_3$ is suffering from Scrub Typhus.

6. Conclusion

The main goal of this work is to present a Type-II GPBFSS and studied some of its properties. Similarity measure of two Type-II GPBFSS's is discussed and an application of this to medical diagnosis has been shown. In the future direction, we will apply the generalized cubic fuzzy soft sets and generalized spherical fuzzy soft sets theory.

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