Journal of Advances in Mathematics and Computer Science

26(6): 1-20, 2018; Article no.JAMCS.39176

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Existence of Global Attractor for Cahn-Hilliard Perturbed Phase-Field System with Dirichlet Boundary Condition and Regular Potentiel

Mangoubi Jean De Dieu[1*], Daniel Moukoko[1], Dieudonne Ampini[1] and Fidéle Moukamba[1]

[1] *Faculté des Sciences et Techniques, Universit Marien Ngouabi, B.P.69, Brazzaville, Congo.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/39176

Editor(s):

(1) Carlo Bianca, Professor, Laboratoire de Physique Theorique de la Matiere Condensee, Sorbonne Universites, France.

Reviewers:

(1) Sanjib Kumar Datta, University of Kalyani, India.

(2) Seval Catal, Dokuz Eylul University, Turkey.

Complete Peer review History: http://sciencedomain.org/review-history/23715

Received: ***2nd December 2017***

Accepted: ***5th February 2018***

Published: ***19th March 2018***

Original Research Article

Abstract

Aims/ objectives: We are interested in a hyperbolic phase field system of Cahn-Hilliard type, parameterized by ϵ for which the solution is a function defined on $(0;T) \times \Omega$. We show the existence and uniqueness of the solutoin, existence of the global attractor for a hyperbolic phase field system of Cahn-Hilliard type, with homogenous conditions Dirichlet on the boundary, this system is governed by a regular potential, in a bounded and smooth domain. the hyperbolic phase field system of Cahn-Hilliard type is based on a thermomecanical theory of deformable continu. Note that the global attractor is the smallest compact set in the phase space, which is invariant by the semigroup and attracts all bounded sets of initial data, as time goes to infinity. So the global attractor allows to make description of asymptotic behaviour about dynamic system.

Study Design: Propagation study of waves.

Place and Duration of Study: Departement of mathematics (group of research called G.R.A.F.E.D.P), Sciences Faculty and Technical of Marien NGOUABI University PO Box 69, between October 2015 and July 2016.

Corresponding author: E-mail: mmangoubijean@gmail.com

Methodology: To prove the existence of the global attractor to based of the classic methode about the perturbed hyperbolic system, with initial conditions and homogenous conditions Dirichlet on the boundary, we proceed by proving the dissipativity and regularity of the semigroup associated to the system, and we then split the semigroup such that we have the sum of two continuous operators, where the first tends uniformly to zero when the time goes to infinity, and the second is regularizing.
Results: We show the existence of global attractor, about a hyperbolic phase field system of Cahn-Hilliard type, governed by regular potential.
Conclusion: All the procedures explained in the methodology being demonstrated , we can assert the existence of the smallest compact set of the phase space, invariant by the semigroup and which attracts all the bounded sets of initial data from a some time.

Keywords: Cahn-Hilliard phase-field system; dissipativity; global attractor; dirichlet boundary conditions.

2018 Mathematics Subject Classification: 53C25, 83C05, 57N16.

1 Introduction and Setting of the Problem

We recall that the global attractor A is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e. S(t)A = A; $\forall t > 0$) and attracts all bounded sets, of initial data when time goes to infinity. The property of invariance satisfied by the global attractor makes sure of its unicity (when the global attractor exists). It is the smallest closed set which verifies the property of attraction; and it thus appears as a suitable object in view of the study of the asymptotic behaviour of the system. In fact the global attractor is the smallest compact set of the phase space which contains the solution of a dynamic system, when time goes to infinity.

The G. Caginalp phase-field system

$$\frac{\partial u}{\partial t} - \Delta^2 u - \Delta f(u) = -\Delta\theta \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \tag{1.2}$$

has been proposed in [1] as model phase transition processes such as melting-solidification processes and have been studied, e.g.,[2],[3] and [4]; see also, e.g.,([5],[6]). In the above system, u is the order parameter and θ is the (relative) temperature.

These Cahn-Hilliard phase-field system are known as the conserved phase-field system (see [7], [5], [8], [9], [10] and [11]) based on type III heat conduction and with two temperatures (see [12],[13]) in which the authors have proved the existence and the uniqueness of the solution, the existence of global attractor and of exponential attractors with singular or regular potential.

In [14], Mangoubi and al. studied the following Cahn-Hilliard phase-field system

$$\epsilon(-\Delta)\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\frac{\partial \alpha}{\partial t} \tag{1.3}$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial^2 \alpha}{\partial t^2} - \Delta\frac{\partial \alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} \tag{1.4}$$

where $\epsilon > 0$, u is the order parameter and α is the (relative) temperature, the authors have proved the existence and the uniqueness of solution with Dirichlet boundary conditions and a regular potential. The system (1.3)-(1.4) is not dissipative, then to circumvent this difficulty we are forced to add another perturbed term on the equation Cahn-Hilliard (1.3) in order to prove the dissipativity of the below system.

In this paper, we consider the following Cahn-Hilliard perturbed phase-fiel system

$$\epsilon(-\Delta)\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=-\Delta\frac{\partial \alpha}{\partial t} \quad in \quad \mathbf{R}^+\times\Omega \tag{1.5}$$

$$\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial^2 \alpha}{\partial t^2}-\Delta\frac{\partial \alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \quad in \quad \mathbf{R}^+\times\Omega \tag{1.6}$$

$$u|_\Gamma=\alpha|_\Gamma=\Delta u|_\Gamma=0 \tag{1.7}$$

$$u|_{t=0}=u_0,\frac{\partial u}{\partial t}|_{t=0}=u_1,\alpha|_{t=0}=\alpha_0,\frac{\partial \alpha}{\partial t}|_{t=0}=\alpha_1, \tag{1.8}$$

as one perturbed Cahn-Hilliard phase-field system (1.1)-(1.2) with $\epsilon > 0$. Ω is a bounded and regular domain of $\mathbb{R}^n$ $n = 2$ or 3 and f is a nonlinear regular potential.

The hyperbolic system has been extensively studied for Dirichlet boundary conditions and regular or singular potential (see [15], [16], [17]). Whose certain have to end at the existence of global attractor and at the existence of exponential attractors (see [18], [19] and [20]).

In this paper we prove the existence and the uniqueness of solutions and the existence of global attractor of the problem (1.5)-(1.8). We consider here only one type boundary condition, namely, Dirichlet. Furthermore we consider the regular potential $f(s) = s^3 - s$ which satisfies the following properties

$$f \text{ is of class } C^2; f(0)=0, \tag{1.9}$$

$$-c_0 \leqslant f'(s), \quad c_0>0, \quad \forall s\in\mathbb{R}, \tag{1.10}$$

$$-c_1 \leqslant F(s) \leqslant f(s)s+c_2, \quad c_1, \quad c_2 \geqslant 0, \quad \forall s\in\mathbb{R} \tag{1.11}$$

with

$$F(s)=\int_0^s f(\tau)d\tau,$$

.

2 Notations

We denote by $\|.\|$ the usual L^2-norm (with associated product scalar $(.,.)$) and $\|.\|_{-1} = \|(-\Delta)^{\frac{-1}{2}}.\|$, where $-\Delta$ denotes the minus Laplace operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denote the norm of Banach space X and c_p is the constant of Poincare.

Throughout this paper, the same letters C_1, C_2 and C_3 denote (generally positive) constants which may change from line to line, or even in the same line. In what follows, the Poincare, Holder and Young inequality are extensively used, without further referring to them.

3 A Priory Estimates

We multiply (1.5) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω. We have

$$\left(\epsilon\frac{\partial^2 u}{\partial t^2},\frac{\partial u}{\partial t}\right)+\left(\epsilon\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+\left((-\Delta)^{-1}\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+\left(-\Delta u,\frac{\partial u}{\partial t}\right)+\left(f(u),\frac{\partial u}{\partial t}\right)=\left(\frac{\partial \alpha}{\partial t},\frac{\partial u}{\partial t}\right)$$

which implies

$$\frac{d}{dt}\left(\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\bigtriangledown u\|^2+2\int_\Omega F(u)dx\right)+2\epsilon\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|_{-1}^2=2\left(\frac{\partial \alpha}{\partial t},\frac{\partial u}{\partial t}\right). \tag{3.1}$$

Now multiply (1.6) by $\frac{\partial \alpha}{\partial t}$ and integrate over Ω. We obtain

$$\left(\frac{\partial^2 \alpha}{\partial t^2},\frac{\partial \alpha}{\partial t}\right)+\left(-\Delta\frac{\partial^2 \alpha}{\partial t^2},\frac{\partial \alpha}{\partial t}\right)+\left(-\Delta\frac{\partial \alpha}{\partial t},\frac{\partial \alpha}{\partial t}\right)+\left(-\Delta\alpha,\frac{\partial \alpha}{\partial t}\right)=-\left(\frac{\partial u}{\partial t},\frac{\partial \alpha}{\partial t}\right)$$

which yields

$$\frac{d}{dt}\left(\|\frac{\partial \alpha}{\partial t}\|^2+\|\bigtriangledown\frac{\partial \alpha}{\partial t}\|^2+\|\nabla\alpha\|^2\right)+2\|\nabla\frac{\partial \alpha}{\partial t}\|^2=-2\left(\frac{\partial u}{\partial t},\frac{\partial \alpha}{\partial t}\right). \tag{3.2}$$

Summing (3.1) and (3.2), we find

$$\frac{dE_1}{dt}+2\epsilon\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|_{-1}^2+2\|\bigtriangledown\frac{\partial \alpha}{\partial t}\|^2=0, \tag{3.3}$$

where

$$E_1=\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\bigtriangledown u\|^2+2\int_\Omega F(u)dx+\|\frac{\partial \alpha}{\partial t}\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\bigtriangledown\alpha\|^2. \tag{3.4}$$

and satisfies

$$E_1\geqslant C\left(\|\bigtriangledown u\|^2+\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\bigtriangledown\alpha\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2\right)+C',\quad C>0. \tag{3.5}$$

We conclude that $u,\alpha\in L^\infty\left(R^+;H_0^1(\Omega)\right)$,

$$\frac{\partial u}{\partial t}\in L^\infty\left(R^+;L^2(\Omega)\right)\cap L^2\left(0,T;L^2(\Omega)\right)$$

and

$$\frac{\partial \alpha}{\partial t}\in L^\infty\left(R^+;H_0^1(\Omega)\right)\cap L^2\left(0,T;H_0^1(\Omega)\right)\ \forall T>0.$$

Multiply (1.6) by $\frac{\partial^2\alpha}{\partial t^2}$ and integrate over Ω. We get

$$\begin{aligned}
2\|\frac{\partial^2 \alpha}{\partial t^2}\|^2+2\|\nabla\frac{\partial^2 \alpha}{\partial t^2}\|^2+\frac{d}{dt}\|\nabla\frac{\partial \alpha}{\partial t}\|^2 &= -2(\frac{\partial u}{\partial t},\frac{\partial^2 \alpha}{\partial t^2})-2(\nabla\alpha,\nabla\frac{\partial \alpha}{\partial t})\\
&\leq 2\|\frac{\partial^2 \alpha}{\partial t^2}\|\|\frac{\partial u}{\partial t}\|+2\|\nabla\alpha\|\|\nabla\frac{\partial^2 \alpha}{\partial t^2}\|\\
\frac{d}{dt}\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\frac{\partial^2 \alpha}{\partial t^2}\|^2+\|\nabla\frac{\partial^2 \alpha}{\partial t^2}\|^2 &\leq \|\frac{\partial u}{\partial t}\|^2+\|\nabla\alpha\|^2.
\end{aligned}$$

Then $\frac{\partial^2\alpha}{\partial t^2} \in L^2(0,T;H_0^1(\Omega))$. In the following section, we have three main results: existence and uniqueness theorems and the existence of solution with more regularity.

4 Existence and Uniqueness of Solutions

Theorem 4.1. *(Existence) We assume that* $(u_0,u_1,\alpha_0,\alpha_1) \in H_0^1(\Omega)\times L^2(\Omega)\times\left(H_0^1(\Omega)\right)^2$*. Then, the system* $(1.5)-(1.8)$ *possesses at least one solution* (u,α) *such that* $u,\alpha \in L^\infty\left(\mathbf{R}^+;H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(\mathbf{R}^+;L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$, $\frac{\partial\alpha}{\partial t} \in L^\infty\left(\mathbf{R}^+;H_0^1(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$ *and* $\frac{\partial^2\alpha}{\partial t^2} \in L^2\left(0,T;H_0^1(\Omega)\right)$, $\forall T>0$.

The proof is based on priory estimates obtained in the previous section and on a standard Galerkin scheme.

Theorem 4.2. *(Uniqueness) Let the assumptions of Theorem 4.1 hold. Then, the system* $(1.5)-(1.8)$ *possesses a unique solution* (u,α) *such that* $u,\alpha \in L^\infty\left(\mathbf{R}^+;H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(\mathbf{R}^+;L^2(\Omega)\right) \cap L^2\left(0,T;L^2(\Omega)\right)$, $\frac{\partial\alpha}{\partial t} \in L^\infty\left(\mathbf{R}^+;H_0^1(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right)$ *and* $\frac{\partial^2\alpha}{\partial t^2} \in L^2\left(0,T;H_0^1(\Omega)\right)$, $\forall T>0$.

Proof. Let $\left(u^{(1)},\alpha^{(1)}\right)$ and $\left(u^{(2)},\alpha^{(2)}\right)$ be two solutions of the system $(1.5)-(1.8)$ with initial data $\left(u_0^{(1)},u_1^{(1)},\alpha_0^{(1)},\alpha_1^{(1)}\right)$ and $\left(u_0^{(2)},u_1^{(2)},\alpha_0^{(2)},\alpha_1^{(2)}\right) \in H_0^1(\Omega)\times L^2(\Omega)\times\left(H_0^1(\Omega)\right)^2$, respectively. We set $u=u^{(1)}-u^{(2)}$ and $\alpha=\alpha^{(1)}-\alpha^{(2)}$, then (u,α) is one solution of the following system

$$\epsilon(-\Delta)\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)+\frac{\partial u}{\partial t}+\Delta^2 u-\Delta\left(f(u^1)-f(u^2)\right)=-\Delta\frac{\partial\alpha}{\partial t} \tag{4.1}$$

$$\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial^2\alpha}{\partial t^2}-\Delta\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t} \tag{4.2}$$

$$u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=\alpha|_{\partial\Omega}=0$$

$$u|_{t=0}=u_0=u_0^{(1)}-u_0^{(2)};\quad \frac{\partial u}{\partial t}|_{t=0}=u_1=u_1^{(1)}-u_1^{(2)}$$

$$\alpha|_{t=0}=\alpha_0=\alpha_0^{(1)}-\alpha_0^{(2)};\quad \frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1=\alpha_1^{(1)}-\alpha_1^{(2)}.$$

We multiply (4.1) by $(-\Delta)^{-1}\dfrac{\partial u}{\partial t}$ and integrate over Ω. We obtain

$$\frac{d}{dt}\left(\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\bigtriangledown u\|^2\right)+2\epsilon\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|_{-1}^2+2\left(f(u^{(1)})-f(u^{(2)}),\frac{\partial u}{\partial t}\right)=2\left(\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}\right). \tag{4.3}$$

Multiplying (4.2) by $\dfrac{\partial\alpha}{\partial t}$ and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\frac{\partial\alpha}{\partial t}\|^2+\|\bigtriangledown\frac{\partial\alpha}{\partial t}\|^2+\|\nabla\alpha\|^2\right)+2\|\nabla\frac{\partial\alpha}{\partial t}\|^2=-2\left(\frac{\partial u}{\partial t},\frac{\partial\alpha}{\partial t}\right). \tag{4.4}$$

Summing (4.3) and (4.4), then we obtain

$$\frac{dE_2}{dt} + 2\epsilon\|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\| \bigtriangledown \frac{\partial \alpha}{\partial t}\|^2 = -2\left(f(u^{(1)}) - f(u^{(2)}), \frac{\partial u}{\partial t}\right) \tag{4.5}$$

where

$$E_2 = \epsilon\|\frac{\partial u}{\partial t}\|^2 + \| \bigtriangledown u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \| \bigtriangledown \alpha\|^2.$$

We know that

$$f(u^{(1)}) - f(u^{(2)}) = l(t)u$$

with

$$\begin{aligned} l(t) &= \int_0^1 f'(u^{(2)} + s(u^{(1)} - u^{(2)}))ds \\ &\leqslant 3\int_0^1 (su^{(1)} + (1-s)\,u^{(2)})^2 ds, \\ &\leqslant 3\int_0^1 (|u^{(1)}| + |u^{(2)}| + 1)^2 ds \\ &\leqslant C\int_0^1 (|u^{(1)}|^2 + |u^{(2)}|^2 + 1)ds \\ &\leqslant C(|u^{(1)}|^2 + |u^{(2)}|^2 + 1), \end{aligned}$$

Then

$$\begin{aligned} \int_\Omega (f(u^{(1)}) - f(u^{(2)}))\frac{\partial u}{\partial t}dx &= \int_\Omega l(t)|u||\frac{\partial u}{\partial t}|dx \\ &\le C\int_\Omega \left(|u^{(1)}|^2 + |u^{(2)}|^2 + 1\right)|u||\frac{\partial u}{\partial t}|dx, \\ &\leqslant C\left(\|u^{(1)}\|_{L^6}^2 + \|u^{(2)}\|_{L^6}^2 + 1\right)\|u\|_{L^6}\|\frac{\partial u}{\partial t}\| \\ &\leqslant C\left(\|u^{(1)}\|_{H^1}^2 + \|u^{(2)}\|_{H^1}^2 + 1\right)\|u\|_{H^1}\|\frac{\partial u}{\partial t}\| \\ &\leqslant C\left(\|u\|_{H^1}\|\frac{\partial u}{\partial t}\|\right) \\ &\leqslant K\left(\|u\|_{H^1}^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2\right), \quad K > 0. \end{aligned} \tag{4.6}$$

Inserting (4.6) into (4.5), we have

$$\frac{dE_2}{dt} + 2\epsilon\|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|_{-1}^2 + 2\| \bigtriangledown \frac{\partial \alpha}{\partial t}\|^2 \le K(\|\nabla u\|^2 + \epsilon\|\frac{\partial u}{\partial t}\|^2).$$

Applying Gronwalls lemma, we obtain $\forall\ t \in [0;\ T]$

$$E_2(t) + 2\int_0^t (\epsilon\|\frac{\partial u}{\partial t}(\tau)\|^2 + \|\frac{\partial u}{\partial t}(\tau)\|_{-1}^2 + \|\nabla\frac{\partial \alpha}{\partial t}(\tau)\|^2)d\tau \le E_2(0)e^{KT}. \tag{4.7}$$

We deduce the continuous dependence of the solution relative to the initial conditions, hence the uniqueness of the solution.

The existence and the uniqueness of the solution of problem (1.5)-(1.8) being proved in a larger space, we will seek the existence of solution with more regularity. □

Theorem 4.3. *We assume that* $(u_0, u_1, \alpha_0, \alpha_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \times \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^2$. *Then the system (1.5)-(1.8) possesses a unique solution* (u, α) *such that* $u, \alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$, $\frac{\partial \alpha}{\partial t} \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap L^2\left(0, T; H^2(\Omega)\right)$, $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$ *and* $\frac{\partial^2 u}{\partial t^2} \in L^2\left(0, T; L^2(\Omega)\right)$, $\forall T > 0$.

Proof. Owing to the theorems (4.1) and (4.2), the system (1.5) − (1.8) possesses a unique solution (u, α) such that $u, \alpha \in L^\infty\left(R^+; H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(\mathbf{R}^+; L^2(\Omega)\right) \cap L^2\left(0, T; L^2(\Omega)\right)$, $\frac{\partial \alpha}{\partial t} \in L^\infty\left(\mathbf{R}^+; H_0^1(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$ *and* $\frac{\partial^2 \alpha}{\partial t^2} \in L^2\left(0, T; H_0^1(\Omega)\right)$, $\forall T > 0$.

Multiply (1.5) by $\frac{\partial u}{\partial t}$ and integrate over Ω. We have

$$\frac{d}{dt}\left(\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2\right) + 2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|^2 = 2\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\frac{\partial u}{\partial t}\right) - 2\left(\nabla f(u), \nabla\frac{\partial u}{\partial t}\right). \quad (4.8)$$

Multiplying (1.6) by $-\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\alpha\|^2\right) + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 = -2\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\frac{\partial u}{\partial t}\right). \quad (4.9)$$

Summing (4.8) and (4.9), we obtain

$$\frac{dE_3}{dt} + 2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial u}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 \quad = \quad -2\left(f'(u)\nabla u, \nabla\frac{\partial u}{\partial t}\right), \quad (4.10)$$

where

$$E_3 = \epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\Delta\alpha\|^2.$$

Thanks to use $f'(s)$ and the fact that $u \in L^\infty\left(\mathbf{R}_+; H_0^1(\Omega)\right)$, we find the following estimate

$$\begin{aligned}
2\int_\Omega |f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx &= \int_\Omega |3u^2 - 1||\nabla u||\nabla\frac{\partial u}{\partial t}|dx \\
&\leqslant 3\int_\Omega (|u|^2 + 1)|\nabla u||\nabla\frac{\partial u}{\partial t}|dx \\
&\leqslant C\left(\|u\|_{L^6}^2 + 1\right)\|\nabla u\|_{L^6}\|\nabla\frac{\partial u}{\partial t}\| \\
&\leqslant C\left(\|u\|_{H^1}^2 + 1\right)\|\Delta u\|\|\nabla\frac{\partial u}{\partial t}\| \\
&\leqslant C\|\Delta u\|\|\nabla\frac{\partial u}{\partial t}\|, \\
&\leqslant K\left(\|\Delta u\|^2 + \epsilon\|\nabla\frac{\partial u}{\partial t}\|^2\right), \ K > 0. \quad (4.11)
\end{aligned}$$

Inserting (4.11) into (4.10) and applying the Gronwall's lemma, we deduce that $u, \alpha \in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^\infty\left(0, T; H_0^1(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$ and

$\frac{\partial\alpha}{\partial t} \in L^\infty(0,T;H^2(\Omega)\cap H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)),\ \forall\, T>0.$

Multiplying (1.5) by $(-\Delta)^{-1}\dfrac{\partial^2 u}{\partial t^2}$ and integrating over Ω, we obtain

$$\begin{aligned}2\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2+\epsilon\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2+\frac{d}{dt}\|\frac{\partial u}{\partial t}\|_{-1}^2 &= 2\left(\frac{\partial\alpha}{\partial t},\frac{\partial^2 u}{\partial t^2}\right)+2(\Delta u,\frac{\partial^2 u}{\partial t^2})-2\left(f(u),\frac{\partial^2 u}{\partial t^2}\right),\\ &\le 2\|\frac{\partial\alpha}{\partial t}\|\|\frac{\partial^2 u}{\partial t^2}\|+2\|\Delta u\|\|\frac{\partial^2 u}{\partial t^2}\|\\ &+2\int_\Omega |f(u)||\frac{\partial^2 u}{\partial t^2}|dx. \end{aligned}\tag{4.12}$$

Thanks to use $f(s)$ and the fact that $u\in H^2(\Omega)\subset L^\infty(\Omega)$, we get

$$\begin{aligned}\int_\Omega |f(u)||\frac{\partial^2 u}{\partial t^2}|dx &\le \left(\|u^2\|_{L^\infty}+1\right)\int_\Omega |u||\frac{\partial^2 u}{\partial t^2}|dx\\ &\le C\int_\Omega |u||\frac{\partial^2 u}{\partial t^2}|dx\\ &\le C\|u\|\|\frac{\partial^2 u}{\partial t^2}\|\\ &\le C\|\nabla u\|\|\frac{\partial^2 u}{\partial t^2}\|\\ &\le C\|\nabla u\|^2+\frac{\epsilon}{3}\|\frac{\partial^2 u}{\partial t^2}\|^2.\end{aligned}$$

Inserting the above estimate into (4.12), we obtain

$$\frac{d}{dt}\left(\epsilon\|\frac{\partial u}{\partial t}\|^2+\|\frac{\partial u}{\partial t}\|_{-1}^2\right)+\epsilon\|\frac{\partial^2 u}{\partial t^2}\|^2+\leqslant C_1(\|\frac{\partial\alpha}{\partial t}\|^2+\|\Delta u\|^2+\|\nabla u\|^2),\ C_1>0,$$

which implies that $\frac{\partial^2 u}{\partial t^2}\in L^2(0,T;L^2(\Omega))$.

Multiplying (1.6) by $-\Delta\frac{\partial^2\alpha}{\partial t^2}$ and integrating over Ω, we find

$$\begin{aligned}2\|\nabla\frac{\partial^2\alpha}{\partial t^2}\|^2+2\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|^2+\frac{d}{dt}\|\Delta\frac{\partial\alpha}{\partial t}\|^2 &\leqslant 2\|\frac{\partial u}{\partial t}\|\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|+2\|\Delta\alpha\|\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|,\\ &\leqslant 2\|\Delta\alpha\|^2+\frac{1}{2}\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|^2+\frac{1}{2}\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|^2+2\|\frac{\partial u}{\partial t}\|^2\\ \frac{d}{dt}\|\Delta\frac{\partial\alpha}{\partial t}\|^2+2\|\nabla\frac{\partial^2\alpha}{\partial t^2}\|^2+\|\Delta\frac{\partial^2\alpha}{\partial t^2}\|^2 &\leqslant 2\|\frac{\partial u}{\partial t}\|^2+2\|\Delta\alpha\|^2\end{aligned}$$

that implies $\frac{\partial^2\alpha}{\partial t^2}\in L^2\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)$. □

5 Dissipativity and Regularity

In this section, we have two main results, the dissipativity and the regularity of the semi-group $\{S_\varepsilon(t)\}_{t\geqslant 0}$ associated to the system (1.5) – (1.8).

We have thanks to the Theorems (4.2) and (4.3) two following respective phase spaces

$$\Phi_1 = H_0^1(\Omega)\times L^2(\Omega)\times (H_0^1(\Omega))^2,$$
$$\Phi_2 = (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times (H^2(\Omega)\cap H_0^1(\Omega))^2,$$

and the two following energy norms

$$\|(u, \frac{\partial u}{\partial t}, \alpha, \frac{\partial \alpha}{\partial t})\|^2_{\Phi_k} = \|u\|^2_{H^k} + \epsilon\|\frac{\partial u}{\partial t}\|^2_{H^{k-1}} + \|\alpha\|^2_{H^k} + \|\frac{\partial \alpha}{\partial t}\|^2_{H^k}, \quad k = 1, 2.$$

We then define the continuous semi-group

$$S_t(\varepsilon) : \Phi_k \longrightarrow \Phi_k \, (u_0, u_1, \alpha_0, \alpha_1) \longmapsto \left(u(t), \frac{\partial u(t)}{\partial t}, \alpha(t), \frac{\partial \alpha(t)}{\partial t} \right), \tag{5.1}$$

where $k = 1, 2$ and (u, α) is the unique solution of the system $(1.5) - (1.8)$ with the initial conditions $(u_0, u_1, \alpha_0, \alpha_1) \in \Phi_k$.

Theorem 5.1. If we assume that (u, α) is the solution of the problem $(1.5) - (1.8)$ with initial data that $(u_0, u_1, \alpha_0, \alpha_1) \in \phi_1$. Then, the solution (u, α) satisfies the following estimate

$$\begin{aligned} &\| \left(u(t), \frac{\partial u}{\partial t}(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t) \right) \|_{\Phi_1} + \int_0^t \|\frac{\partial u(\tau)}{\partial t}\|^2_{-1} e^{-\beta(t-\tau)} d\tau \\ &\leqslant Q \left(\|u_0\|_{H^1}, \|u_1\|, \|\alpha_0\|_{H^1}, \|\alpha_1\|_{H^1} \right) e^{-\beta t} + C \end{aligned} \tag{5.2}$$

where β and C are the positive constants and Q is the monotonic function.

Proof. We multiply (1.5) by $(-\Delta)^{-1}u$ and integrate over Ω. We have

$$\epsilon \left(\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t}, u \right) + \left((-\Delta)^{-1} \frac{\partial u}{\partial t}, u \right) + (-\Delta u, u) + \int_\Omega f(u) u dx = \left(\frac{\partial \alpha}{\partial t}, u \right),$$

Using (1.11) we have

$$\frac{d}{dt} E_4 + 2\|\nabla u\|^2 + \int_\Omega F(u) dx \leqslant \int_\Omega c_2 dx + 2 \left(\frac{\partial \alpha}{\partial t}, u \right) + 2\epsilon \|\frac{\partial u}{\partial t}\|^2,$$

which implies

$$\frac{d}{dt} E_4 + \|\nabla u\|^2 + 2 \int_\Omega F(u) dx \leqslant C + c_p^2 \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2\epsilon \|\frac{\partial u}{\partial t}\|^2, \tag{5.3}$$

where

$$E_4 = \left(2\epsilon(\frac{\partial u}{\partial t}, u) + \|u\|^2_{-1} + \epsilon\|u\|^2 \right).$$

We multiply (1.6) by α, and integrate over Ω. We have

$$\left(\frac{\partial^2 \alpha}{\partial t^2}, \alpha \right) + \left(-\Delta \frac{\partial^2 \alpha}{\partial t^2}, \alpha \right) + \left(-\Delta \frac{\partial \alpha}{\partial t}, \alpha \right) + (-\Delta \alpha, \alpha) = - \left(\frac{\partial u}{\partial t}, \alpha \right),$$

which implies

$$\begin{aligned} \frac{d}{dt} E_5 + 2\|\nabla \alpha\|^2 &\leqslant \|\frac{\partial u}{\partial t}\|_{-1} \|\nabla \alpha\| + 2\|\frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla \frac{\partial \alpha}{\partial t}\|^2 \\ &\leqslant \|\frac{\partial u}{\partial t}\|^2_{-1} + \|\nabla \alpha\|^2 + 2c_p \|\nabla \frac{\partial \alpha}{\partial t}\|^2 + 2\|\nabla \frac{\partial \alpha}{\partial t}\|^2 \\ \frac{d}{dt} E_5 + \|\nabla \alpha\|^2 &\leqslant \|\frac{\partial u}{\partial t}\|^2_{-1} + C \|\nabla \frac{\partial \alpha}{\partial t}\|^2. \end{aligned} \tag{5.4}$$

where

$$E_5 = \left(2(\frac{\partial \alpha}{\partial t}, \alpha) + 2(\nabla \frac{\partial \alpha}{\partial t}, \nabla \alpha) + \|\nabla \alpha\|^2 \right). \tag{5.5}$$

Summing (3.3), γ_1(5.3) and γ_2(5.4) where γ_1 and $\gamma_2 > 0$ are such that

$$\begin{aligned} 1-\gamma_1 &> 0 \\ 2-\gamma_2 &> 0, \\ 1-\gamma_1 c_p^2 - \gamma_2 C &> 0 \end{aligned}$$

we get

$$\frac{d}{dt}E_6 + C_1\|\nabla u\|^2 + C_2\|\frac{\partial u}{\partial t}\|^2 + C_3\|\frac{\partial u}{\partial t}\|_{-1}^2 + C_4\|\nabla\alpha\|^2 + 2\gamma_1\int_\Omega F(u)dx + C_5\|\nabla\frac{\partial\alpha}{\partial t}\|^2 \leqslant C, \quad (5.6)$$

where

$$E_6 = E_1 + \gamma_1 E_4 + \gamma_2 E_5.$$

Moreover, for sufficiently small values of γ_1 and $\gamma_2 > 0$, there exists $C > 0$ such that

$$\begin{aligned} &C^{-1}\left(\epsilon\|\frac{\partial u}{\partial t}(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}(t)\|^2 + \|\nabla\alpha(t)\|^2\right) \leqslant E_6(t) \\ &\leqslant C\left(\epsilon\|\frac{\partial u}{\partial t}(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla\frac{\partial\alpha}{\partial t}(t)\|^2 + \|\nabla\alpha(t)\|^2\right). \quad (5.7) \end{aligned}$$

Thanks to (5.7), (5.6) can be written as

$$\frac{d}{dt}E_6 + \beta E_6 + C\|\frac{\partial u}{\partial t}\|_{-1}^2 \leqslant C, \quad (5.8)$$

where β and C are the positive constants. Applying Gronwall's lemma, thanks to estimate (5.7) we have

$$\begin{aligned} &\|(u(t), \frac{\partial u}{\partial t}(t), \alpha(t), \frac{\partial\alpha}{\partial t}(t))\|_{\Phi_1}^2 + C\int_0^t \|\frac{\partial u(\tau)}{\partial t}\|_{-1}^2 e^{-\beta(t-\tau)}d\tau \\ &\leqslant Q\left(\|u_0\|_{H^1}, \|u_1\|, \|\alpha_0\|_{H^1}, \|\alpha_1\|_{H^1}\right)e^{-\beta t} + C, \quad (5.9) \end{aligned}$$

where β and C are the positive constants and Q is the monotonic function. □

Corollary 5.2. The semi-group of operators $S_\varepsilon(t), t \geqslant 0$ associated to the problem (1.5) – (1.8) is dissipative in Φ_1, it possesses a bounded absorbing set in Φ_1.

This corollary is a straightforward consequence of theorem (5.1).

We denote $B_{R_0}(\epsilon) = \{\left(u, \frac{\partial u}{\partial t}, \alpha, \frac{\partial\alpha}{\partial t}\right) \in \Phi_1 / \|\left(u, \frac{\partial u}{\partial t}, \alpha, \frac{\partial\alpha}{\partial t}\right)\|_{\Phi_1} \leqslant R_0\}$ where R_0 is large enough, a bounded absorbing set of the semi-group $S_\varepsilon(t)$ *in* Φ_1.

Theorem 5.3. Assume that (u, α) the solution of the problem (1.5) – (1.8) with initial data $(u_0, u_1, \alpha_0, \alpha_1,) \in B_{R_0}(\epsilon) \cap \Phi_2$. Then, the solution (u, α) verifies the following estimate

$$\begin{aligned} &\|(u(t), \frac{\partial u}{\partial t}(t), \alpha(t), \frac{\partial\alpha}{\partial t}(t))\|_{\Phi_2}^2 + 2\int_0^t \|\frac{\partial u}{\partial t}(\tau)\|_{-1}^2 e^{-\beta(t-\tau)}d\tau \\ &\leqslant Q\left(\|u_0\|_{H^2}, \|u_1\|_{H^1}, \|\alpha_0\|_{H^2}, \|\alpha_1\|_{H^2}\right)e^{-\beta t} + C, \quad (5.10) \end{aligned}$$

where β and C are the positive constants and Q is the monotonic function.

Proof. Multiply (1.5) by $\frac{\partial u}{\partial t}$ *and integrate over* Ω. *We have*

$$\frac{d}{dt}\left(\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+\|\Delta u\|^2\right)+2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|^2=2\left(\nabla\frac{\partial \alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right)-2\left(-\Delta f(u),\frac{\partial u}{\partial t}\right). \tag{5.11}$$

Multiplying (1.6) by $-\Delta\frac{\partial \alpha}{\partial t}$ and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\alpha\|^2\right)+2\|\Delta\frac{\partial \alpha}{\partial t}\|^2=-2\left(\nabla\frac{\partial \alpha}{\partial t},\nabla\frac{\partial u}{\partial t}\right). \tag{5.12}$$

Now summing (5.11) and (5.12). we obtain

$$\frac{d}{dt}E_7+2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|^2+2\|\Delta\frac{\partial \alpha}{\partial t}\|^2=-2\left(-\Delta f(u),\frac{\partial u}{\partial t}\right). \tag{5.13}$$

We know that

$$\begin{aligned}\left(-\Delta f(u),\frac{\partial u}{\partial t}\right)&=\left(f'(u)\nabla u,\nabla\frac{\partial u}{\partial t}\right) &(5.14)\\ &=\int_\Omega |f'(u)||\nabla u||\nabla\frac{\partial u}{\partial t}|dx\\ &\leqslant\int_\Omega (u^2+1)|\nabla u||\nabla\frac{\partial u}{\partial t}|dx\\ &\leqslant(\|u\|_{L^6}^2+1)\|\nabla u\|_{L^6}\|\nabla\frac{\partial u}{\partial t}\|\\ &\leqslant(\|u\|_{H^1}^2+1)\|\Delta u\|\|\nabla\frac{\partial u}{\partial t}\|\\ &\leqslant C\|\Delta u\|\|\nabla\frac{\partial u}{\partial t}\|\\ &\leqslant C\|\Delta u\|^2+\frac{\epsilon}{2}\|\nabla\frac{\partial u}{\partial t}\|^2. &(5.15)\end{aligned}$$

Inserting (5.15) into (5.13) we obtain

$$\frac{d}{dt}E_7+\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial u}{\partial t}\|^2+2\|\Delta\frac{\partial \alpha}{\partial t}\|^2\leqslant C\|\Delta u\|^2, \tag{5.16}$$

where

$$E_7=\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+\|\Delta u\|^2+\|\nabla\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\frac{\partial \alpha}{\partial t}\|^2+\|\Delta\alpha\|^2.$$

Multiply (1.5) by u and integrate over Ω. we obtain

$$\epsilon\left((-\Delta)(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}),u\right)+\left(\frac{\partial u}{\partial t},u\right)+\left(\Delta^2 u,u\right)+(-\Delta f(u),u)=\left(-\Delta\frac{\partial \alpha}{\partial t},u\right)$$

which implies

$$\epsilon\left((-\Delta)(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}),u\right)+\left(\frac{\partial u}{\partial t},u\right)+\|\Delta u\|^2+\left(f'(u)\nabla u,\nabla u\right)=\left(\frac{\partial \alpha}{\partial t},-\Delta u\right). \tag{5.17}$$

$$\begin{aligned}\frac{d}{dt}E_8 + 2\|\Delta u\|^2 &\leqslant 2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + \|\Delta u\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + c_0\|\nabla u\|^2 \\ \frac{d}{dt}E_8 + \|\Delta u\|^2 &\leqslant 2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2 + 2c_p\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + c_0\|\nabla u\|^2. \end{aligned} \tag{5.18}$$

where

$$E_8 = 2\epsilon(\nabla\frac{\partial u}{\partial t}, \nabla u) + \epsilon\|\nabla u\|^2 + \|u\|^2.$$

We multiply (1.6) by $-\Delta\alpha$ and integrate over Ω. We find

$$\begin{aligned}\frac{d}{dt}\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\alpha\right) + \frac{d}{dt}\left(\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha\right) + \frac{1}{2}\frac{d}{dt}\|\Delta\alpha\|^2 + \|\Delta\alpha\|^2 &\leqslant \frac{1}{2}\|\frac{\partial u}{\partial t}\|^2 + \frac{1}{2}\|\Delta\alpha\|^2 \\ &+ \|\Delta\frac{\partial \alpha}{\partial t}\|^2 + \|\nabla\frac{\partial \alpha}{\partial t}\|^2 \\ \frac{d}{dt}\left(2(\nabla\frac{\partial \alpha}{\partial t}, \nabla\alpha) + 2(\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha) + \|\Delta\alpha\|^2\right) + \|\Delta\alpha\|^2 &\leqslant \|\frac{\partial u}{\partial t}\|^2 + 2\|\Delta\frac{\partial \alpha}{\partial t}\|^2 \\ &+ 2\|\nabla\frac{\partial \alpha}{\partial t}\|^2. \end{aligned} \tag{5.19}$$

Now summing (5.6), γ_3(5.16), γ_4(5.18) and γ_5(5.19) where $\gamma_3,\ \gamma_4$ and $\gamma_5 > 0$ are such that

$$\begin{aligned} C_1 - \gamma_4 c_0 &> 0 \\ C_2 + 2\gamma_3 - \gamma_5 &> 0 \\ C_5 - 2\gamma_4 c_p - 2\gamma_5 &> 0, \\ \gamma_3 - \gamma_5 &> 0 \\ \gamma_3 - 2\gamma_4 &> 0 \end{aligned}$$

we find

$$\begin{aligned}&\frac{d}{dt}E_9 + C_1\|\Delta u\|^2 + C_2\|\nabla\frac{\partial u}{\partial t}\|^2 + C_3\|\frac{\partial u}{\partial t}\|^2 + C_4\|\Delta\frac{\partial \alpha}{\partial t}\|^2 + C_5\|\Delta\alpha\|^2 \\ &+ C_6\|\nabla\frac{\partial \alpha}{\partial t}\|^2 + C_7\|\nabla\alpha\|^2 + C_8\int_\Omega F(u)dx + C_9\|\frac{\partial u}{\partial t}\|^2_{-1} \leqslant C, \end{aligned} \tag{5.20}$$

where

$$E_9 = E_6 + \gamma_3 E_7 + \gamma_4 E_8 + \gamma_5\left(2(\nabla\frac{\partial \alpha}{\partial t}, \nabla\alpha) + 2(\Delta\frac{\partial \alpha}{\partial t}, \Delta\alpha) + \|\Delta\alpha\|^2\right).$$

For sufficiently small values of γ_4 and $\gamma_5 > 0$, there exists $C > 0$ such that

$$\begin{aligned}&C^{-1}\left(\epsilon\|\nabla\frac{\partial u}{\partial t}(t)\|^2 + \|\Delta u(t)\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}(t)\|^2 + \|\Delta\alpha(t)\|^2\right) \leqslant E_9(t) \\ &\leqslant C\left(\epsilon\|\nabla\frac{\partial u}{\partial t}(t)\|^2 + \|\Delta u(t)\|^2 + \|\Delta\frac{\partial \alpha}{\partial t}(t)\|^2 + \|\Delta\alpha(t)\|^2\right). \end{aligned} \tag{5.21}$$

We deduce from the above estimate and (5.20) the following estimate

$$\frac{d}{dt}E_9 + \beta E_9 + C_1\|\frac{\partial u}{\partial t}\|_{-1}^2 \leqslant C$$

Applying Gronwall's lemma, we obtain

$$\|(u(t), \frac{\partial u}{\partial t}(t), \alpha(t), \frac{\partial \alpha}{\partial t}(t))\|_{\Phi_2}^2 + \int_0^t (\|\frac{\partial u}{\partial t}(\tau)\|_{-1}^2)e^{-\beta(t-\tau)}d\tau$$
$$\leqslant Q\left(\|u_0\|_{H^2}, \|u_1\|_{H^1}, \|\alpha_0\|_{H^2}, \|\alpha_1\|_{H^2}\right)e^{-\beta t} + C \tag{5.22}$$

where β and C are the positive constants and Q is the monotonic function. □

Corollary 5.4. *The semigroup of operators $S_\varepsilon(t), t \geqslant 0$ associated to the problem* $(1.5)-(1.8)$ *is dissipative in Φ_2, i.e, it possesses a bounded absorbing set in Φ_2.*

This corollary is a straightforward consequence of theorem (5.3).

6 Existence of Global Attractor

Theorem 6.1. *The semi-group $S_\epsilon(t),\ t \geqslant 0$ defined from Φ_1 into Φ_1 and associated to the problem* $(1.5)-(1.8)$ *possesses the global attractor $\mathcal{A}_\epsilon$ which is compact in Φ_1.*

Proof. We have already proved the dissipativity of the semi-group $\{S_\epsilon(t)\}_{t\geqslant 0}$ associated to the problem $(1.5)-(1.8)$. It remains to split the semi-group $S_\epsilon(t)$ as the sum of two continuous operators $S_\epsilon^1(t)$ and $S_\epsilon^2(t)$, such that the solution (u, α) with initial condition belonging to $B_{R_0} \cap \Phi_2$ can be written as following

$$(u, \alpha) = (v, \eta) + (\omega, \xi)$$

with

$$S_t^1(\epsilon)\left(u(0), \frac{\partial u}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0)\right) = \left(v(t), \frac{\partial v}{\partial t}(t), \eta(t), \frac{\partial \eta}{\partial t}(t)\right)$$

and

$$S_t^2(\epsilon)\left(0,0,0,0\right) = \left(\omega(t), \frac{\partial \omega}{\partial t}(t), \xi(t), \frac{\partial \xi}{\partial t}(t)\right)$$

□

where $S_t^1(\epsilon)$ is the solving operator associated to the linear and perturbed system

$$\epsilon(-\Delta)(\frac{\partial^2 v}{\partial t^2} + \frac{\partial v}{\partial t}) + \frac{\partial v}{\partial t} + \Delta^2 v = -\Delta\frac{\partial \eta}{\partial t} \tag{6.1}$$

$$\frac{\partial^2 \eta}{\partial t^2} - \Delta\frac{\partial^2 \eta}{\partial t^2} - \Delta\frac{\partial \eta}{\partial t} - \Delta\eta = -\frac{\partial v}{\partial t}, \tag{6.2}$$

$$v|_{\partial\Omega} = \eta|_{\partial\Omega} = \Delta v|_{\partial\Omega} = 0,$$

$$v|_{t=0} = u_0,\ \frac{\partial v}{\partial t}|_{t=0} = u_1,\ \eta|_{t=0} = \alpha_0, \frac{\partial \eta}{\partial t}|_{t=0} = \alpha_1,$$

$S_t^2(\epsilon)$ is the solving operator associated to the nonlinear and perturbed system

$$\epsilon(-\Delta)(\frac{\partial^2\omega}{\partial t^2}+\frac{\partial\omega}{\partial t})+\frac{\partial\omega}{\partial t}-\Delta^2\omega-\Delta f(u)=-\Delta\frac{\partial\xi}{\partial t} \tag{6.3}$$

$$\frac{\partial^2\xi}{\partial t^2}-\Delta\frac{\partial^2\xi}{\partial t^2}-\Delta\frac{\partial\xi}{\partial t}-\Delta\xi=-\frac{\partial\omega}{\partial t}, \tag{6.4}$$

$$\omega|_{\partial\Omega}=\xi|_{\partial\Omega}=\Delta\omega|_{\partial\Omega}=0,$$

$$\omega|_{t=0}=\frac{\partial\omega}{\partial t}|_{t=0}=0,\ \xi|_{t=0}=\frac{\partial\xi}{\partial t}|_{t=0}=0$$

and to show that the operator $S_\varepsilon^1(t)$ uniformly converges to 0 over all bounded subset of Φ_1 and $S_\varepsilon^2(t)$ is regularizing on Φ_2, when the time tends to infinity.

Multiply (6.1) by $(-\Delta)^{-1}\,\frac{\partial v}{\partial t}$ and integrate over Ω. We have

$$\epsilon(\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t},\frac{\partial v}{\partial t})+\left((-\Delta)^{-1}\,\frac{\partial v}{\partial t},\frac{\partial v}{\partial t}\right)+\left(-\Delta v,\frac{\partial v}{\partial t}\right)=\left(\frac{\partial\eta}{\partial t},\frac{\partial v}{\partial t}\right)$$

which yields

$$\frac{d}{dt}\left(\epsilon\|\frac{\partial v}{\partial t}\|^2+\|\nabla v\|^2\right)+2\epsilon\|\frac{\partial v}{\partial t}\|^2+2\|\frac{\partial v}{\partial t}\|_{-1}^2=2\left(\frac{\partial\eta}{\partial t},\frac{\partial v}{\partial t}\right). \tag{6.5}$$

Multiply (6.2) by $\frac{\partial\eta}{\partial t}$ and integrate over Ω. We obtain

$$\left(\frac{\partial^2\eta}{\partial t^2},\frac{\partial\eta}{\partial t}\right)+\left(-\Delta\frac{\partial^2\eta}{\partial t^2},\frac{\partial\eta}{\partial t}\right)+\left(-\Delta\frac{\partial\eta}{\partial t},\frac{\partial\eta}{\partial t}\right)+\left(-\Delta\eta,\frac{\partial\eta}{\partial t}\right)=-\left(\frac{\partial v}{\partial t},\frac{\partial\eta}{\partial t}\right) \tag{6.6}$$

which implies

$$\frac{d}{dt}\left(\|\frac{\partial\eta}{\partial t}\|^2+\|\bigtriangledown\frac{\partial\eta}{\partial t}\|^2+\|\nabla\eta\|^2\right)+2\|\nabla\frac{\partial\eta}{\partial t}\|^2=-2\left(\frac{\partial v}{\partial t},\frac{\partial\eta}{\partial t}\right). \tag{6.7}$$

We sum (6.5) and (6.7), we find

$$\frac{d}{dt}E_{10}+2\epsilon\|\frac{\partial v}{\partial t}\|^2+2\|\frac{\partial v}{\partial t}\|_{-1}^2+2\|\bigtriangledown\frac{\partial\eta}{\partial t}\|^2=0 \tag{6.8}$$

where

$$E_{10}=\epsilon\|\frac{\partial v}{\partial t}\|^2+\|\bigtriangledown v\|^2+\|\frac{\partial\eta}{\partial t}\|^2+\|\nabla\frac{\partial\eta}{\partial t}\|^2+\|\bigtriangledown\eta\|^2.$$

Multiply (6.1) by $(-\Delta)^{-1}v$ and integrate over Ω, we have

$$\epsilon\left(\frac{\partial^2 v}{\partial t^2}+\frac{\partial v}{\partial t},v\right)+\left((-\Delta)^{-1}\frac{\partial v}{\partial t},v\right)+(-\Delta v,v)=(\frac{\partial\eta}{\partial t},v)$$

which implies

$$\frac{d}{dt}\left(2\epsilon\left(\frac{\partial v}{\partial t},v\right)+\epsilon\|v\|^2+\|v\|_{-1}^2\right)+\|\nabla v\|^2\leqslant c_p^2\|\nabla\frac{\partial\eta}{\partial t}\|^2+2\epsilon\|\frac{\partial v}{\partial t}\|^2. \tag{6.9}$$

We multiply (6.2) by η, integrate over Ω. We obtain

$$\left(\frac{\partial^2\eta}{\partial t^2},\eta\right)+\left(-\Delta\frac{\partial^2\eta}{\partial t^2},\eta\right)+\left(-\Delta\frac{\partial\eta}{\partial t},\eta\right)+(-\Delta\eta,\eta)=-\left(\frac{\partial v}{\partial t},\eta\right),$$

which yields

$$\frac{d}{dt}\left(2(\frac{\partial\eta}{\partial t},\eta)+2(\nabla\frac{\partial\eta}{\partial t},\nabla\eta)+\|\nabla\eta\|^2\right)+\|\nabla\eta\|^2\leqslant\|\frac{\partial v}{\partial t}\|_{-1}^2+C\|\nabla\frac{\partial\eta}{\partial t}\|^2. \tag{6.10}$$

Now summing (6.8), γ_6(6.9) and γ_7(6.10) where $\gamma_6,\ \gamma_7>0$ are such that

$$\begin{aligned}1-\gamma_6 &> 0\\ 2-\gamma_7 &> 0,\\ 2-\gamma_6c_p^2-\gamma_7C &> 0\end{aligned}$$

we have the following estimate

$$\frac{d}{dt}E_{11}+C_1\|\frac{\partial v}{\partial t}\|^2+C_2\|\frac{\partial v}{\partial t}\|_{-1}^2+C_3\|\nabla v\|^2+C_4\|\nabla\eta\|^2+C_5\|\nabla\frac{\partial\eta}{\partial t}\|^2\leqslant 0, \tag{6.11}$$

where

$$\begin{aligned}E_{11}=E_{10}&+\gamma_6\left(2\epsilon\left(\frac{\partial v}{\partial t},v\right)+\|v\|_{-1}^2+\|v\|^2\right)\\ &+2\gamma_7\left(\left(\frac{\partial\eta}{\partial t},\eta\right)+\left(\nabla\frac{\partial\eta}{\partial t},\nabla\eta\right)+\|\nabla\eta\|^2\right).\end{aligned}$$

For sufficiently small values of γ_6 and $\gamma_7>0$, there exists $C>0$ such that

$$\begin{aligned}&C^{-1}\left(\epsilon\|\frac{\partial v}{\partial t}(t)\|^2+\|\nabla v(t)\|^2+\|\nabla\frac{\partial\eta}{\partial t}(t)\|^2+\|\nabla\eta(t)\|^2\right)\leqslant E_{11}(t)\\ &\leqslant C\left(\epsilon\|\frac{\partial v}{\partial t}(t)\|^2+\|\nabla v(t)\|^2+\|\nabla\frac{\partial\eta}{\partial t}(t)\|^2+\|\nabla\eta(t)\|^2\right).\end{aligned} \tag{6.12}$$

Thanks to the above estimate, (6.11) can be rewritten as

$$\frac{d}{dt}E_{11}+\beta E_{11}+C\|\frac{\partial v}{\partial t}\|_{-1}^2\leqslant 0 \tag{6.13}$$

where β and C are the positive constants. Applying Gronwall's lemma, thanks to the estimates (6.12), we have

$$\begin{aligned}&\epsilon\|\frac{\partial v}{\partial t}(t)\|^2+\|v(t)\|^2_{H^1}+\|\eta(t)\|^2_{H^1}+\|\frac{\partial \eta}{\partial t}(t)\|^2_{H^1}+\int_0^t\|\frac{\partial v}{\partial t}(\tau)\|^2_{-1}e^{-\beta(t-\tau)}d\tau\\ &\leqslant Q\left(\|(v(0),\frac{\partial v}{\partial t}(0),\alpha(0),\frac{\partial \eta}{\partial t}(0))\|_{\Phi_0}\right)e^{-\beta t}. \qquad (6.14)\end{aligned}$$

So the operator $S^1_t(\epsilon)$ uniformly converges to 0 over all bounded subset of Φ_1 when t tends to infinity.

It remains to prove that $S^2_t(\epsilon)$ is regularizing on Φ_2, when t tends to infinity.

Multiply (6.3) by $\frac{\partial \omega}{\partial t}$ and integrate over Ω. We have

$$\begin{aligned}\frac{d}{dt}\left(\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+\|\Delta\omega\|^2\right)+2\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+2\|\frac{\partial \omega}{\partial t}\|^2 &= 2\left(\nabla\frac{\partial \xi}{\partial t},\nabla\frac{\partial \omega}{\partial t}\right)\\ &- 2\left(f'(u)\nabla u,\nabla\frac{\partial \omega}{\partial t}\right). \qquad (6.15)\end{aligned}$$

Multiplying (6.4) by $-\Delta\frac{\partial \xi}{\partial t}$ and integrating over Ω, we get

$$\frac{d}{dt}\left(\|\nabla\frac{\partial \xi}{\partial t}\|^2+\|\Delta\frac{\partial \xi}{\partial t}\|^2+\|\Delta\xi\|^2\right)+2\|\Delta\frac{\partial \xi}{\partial t}\|^2=-2\left(\nabla\frac{\partial \xi}{\partial t},\nabla\frac{\partial \omega}{\partial t}\right). \qquad (6.16)$$

Summing (6.15) and (6.16), we obtain

$$\frac{d}{dt}E_{12}+\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+2\|\frac{\partial \omega}{\partial t}\|^2+2\|\Delta\frac{\partial \xi}{\partial t}\|^2\leqslant\frac{1}{\epsilon}\|f'(u)\nabla u\|^2, \qquad (6.17)$$

where

$$E_{12}=\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+\|\Delta\omega\|^2+\|\nabla\frac{\partial \xi}{\partial t}\|^2+\|\Delta\frac{\partial \xi}{\partial t}\|^2+\|\Delta\xi\|^2.$$

Multiply (6.3) by ω and integrate over Ω. We get

$$\left(\epsilon(-\Delta)(\frac{\partial^2\omega}{\partial t^2}+\frac{\partial \omega}{\partial t}),\omega\right)+\left(\frac{\partial \omega}{\partial t},\omega\right)+\left(\Delta^2\omega,\omega\right)+(-f(u),\Delta\omega)=\left(\frac{\partial \xi}{\partial t},\Delta\omega\right),$$

which implies

$$\begin{aligned}\frac{d}{dt}E_{13}+2\|\Delta\omega\|^2 &\leqslant 2\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+\frac{1}{2}\|\Delta\omega\|^2+2\|\frac{\partial \xi}{\partial t}\|^2+\frac{1}{2}\|\Delta\omega\|^2+2\|f(u)\|^2\\ &\leqslant 2\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+\|\Delta\omega\|^2+2\|\frac{\partial \xi}{\partial t}\|^2+2\|f(u)\|^2\\ \frac{d}{dt}E_{13}+\|\Delta\omega\|^2 &\leqslant 2\epsilon\|\nabla\frac{\partial \omega}{\partial t}\|^2+c_p^2\|\Delta\frac{\partial \xi}{\partial t}\|^2+C\|f(u)\|^2. \qquad (6.18)\end{aligned}$$

where

$$E_{13} = 2\epsilon\left(\nabla\frac{\partial\omega}{\partial t}, \nabla\omega\right) + \epsilon\|\nabla\omega\|^2 + \|\omega\|^2.$$

We multiply (6.4) by $-\Delta\xi$ and integrate over Ω. We have

$$\begin{aligned}
\frac{d}{dt}\left(\nabla\frac{\partial\xi}{\partial t}, \nabla\xi\right) + \frac{d}{dt}\left(\Delta\frac{\partial\xi}{\partial t}, \Delta\xi\right) + \frac{1}{2}\frac{d}{dt}\|\Delta\xi\|^2 + \|\Delta\xi\|^2 &\leq \|\frac{\partial\omega}{\partial t}\|\|\Delta\xi\| + \|\Delta\frac{\partial\xi}{\partial t}\|^2 + \|\nabla\frac{\partial\xi}{\partial t}\|^2 \\
\frac{d}{dt}\left(2(\nabla\frac{\partial\xi}{\partial t}, \nabla\xi) + 2(\Delta\frac{\partial\xi}{\partial t}, \Delta\xi) + \|\Delta\xi\|^2\right) + 2\|\Delta\xi\|^2 &\leqslant \|\frac{\partial\omega}{\partial t}\|^2 + 2\|\Delta\frac{\partial\xi}{\partial t}\|^2 \\
&+ 2c_p\|\Delta\frac{\partial\xi}{\partial t}\|^2 + \|\Delta\xi\|^2 \\
\frac{d}{dt}\left(2(\nabla\frac{\partial\xi}{\partial t}, \nabla\xi) + 2(\Delta\frac{\partial\xi}{\partial t}, \Delta\xi) + \|\Delta\xi\|^2\right) + \|\Delta\xi\|^2 &\leqslant \|\frac{\partial\omega}{\partial t}\|^2 + C\|\Delta\frac{\partial\xi}{\partial t}\|^2.
\end{aligned}$$

Multiplying (1.5) by $(-\Delta)^{-1}\frac{\partial^2\omega}{\partial t^2}$ and integrating over Ω, we obtain

$$\begin{aligned}
2\epsilon\|\frac{\partial^2\omega}{\partial t^2}\|^2 + \epsilon\frac{d}{dt}\|\frac{\partial\omega}{\partial t}\|^2 + \frac{d}{dt}\|\frac{\partial\omega}{\partial t}\|_{-1}^2 &= 2\left(\frac{\partial\xi}{\partial t}, \frac{\partial^2\omega}{\partial t^2}\right) + 2(\Delta\omega, \frac{\partial^2\omega}{\partial t^2}) - 2\left(f(u), \frac{\partial^2\omega}{\partial t^2}\right) \\
&\leq 2\|\frac{\partial\xi}{\partial t}\|\|\frac{\partial^2\omega}{\partial t^2}\| + 2\|\Delta\omega\|\|\frac{\partial^2\omega}{\partial t^2}\| \\
&+ 2\int_\Omega |f(u)||\frac{\partial^2\omega}{\partial t^2}|dx \\
&\leq C_1\|\Delta\frac{\partial\xi}{\partial t}\|^2 + \frac{\epsilon}{3}\|\frac{\partial^2\omega}{\partial t^2}\|^2 + C_2\|\Delta\omega\|^2 + \frac{\epsilon}{3}\|\frac{\partial^2\omega}{\partial t^2}\|^2 \\
&+ C_3\|f(u)\|^2 + \frac{\epsilon}{3}\|\frac{\partial^2\omega}{\partial t^2}\|^2 \\
\frac{d}{dt}\left(\epsilon\|\frac{\partial\omega}{\partial t}\|^2 + \|\frac{\partial\omega}{\partial t}\|_{-1}^2\right) + \epsilon\|\frac{\partial^2\omega}{\partial t^2}\|^2 &\leq C_1\|\Delta\frac{\partial\xi}{\partial t}\|^2 + C_2\|\Delta\omega\|^2 + C_3\|f(u)\|^2. \qquad (6.19)
\end{aligned}$$

Now add γ_8(6.17), γ_9(6.18), γ_{10}(6.19) and γ_{11}(6.19) where γ_7 γ_8 γ_9 and $\gamma_{10} > 0$ are such that

$$\begin{aligned}
\gamma_8 - 2\gamma_9 - \gamma_{10}c_p &> 0 \\
\gamma_8 - \gamma_9 c_p - \gamma_{10}c_p - \gamma_{11}C' &> 0, \\
\gamma_9 - \gamma_{11} &> 0
\end{aligned}$$

We deduce the following estimates

$$\frac{d}{dt}E_{14} \leqslant C_2\|f(u)\|^2 + C_3\|f'(u)\nabla u\|^2 \qquad (6.20)$$

where

$$\begin{aligned}
E_{14} &= \gamma_8 E_{12} + \gamma_9\left(2\epsilon\left(\nabla\frac{\partial\omega}{\partial t}, \nabla\omega\right) + \|\nabla\omega\|^2 + \|\omega\|^2\right) \\
&+ \gamma_{10}\left(2\left(\nabla\frac{\partial\xi}{\partial t}, \nabla\xi\right) + 2\left(\Delta\frac{\partial\xi}{\partial t}, \Delta\xi\right) + \|\Delta\xi\|^2\right) \\
&+ \gamma_{11}\left(\|\frac{\partial\omega}{\partial t}\|^2 + \|\frac{\partial\omega}{\partial t}\|_{-1}^2\right). \qquad (6.21)
\end{aligned}$$

For sufficiently small values of γ_9 and $\gamma_{10} > 0$, there exists $C > 0$ such that

$$C^{-1}\left(\epsilon\|\nabla\frac{\partial\omega}{\partial t}\|^2 + \|\Delta\omega\|^2 + \|\Delta\frac{\partial\xi}{\partial t}\|^2 + \|\Delta\xi\|^2\right) \leqslant E_{14}(t)$$

$$C\left(\epsilon\|\nabla\frac{\partial\omega}{\partial t}\|^2 + \|\Delta\omega\|^2 + \|\Delta\frac{\partial\xi}{\partial t}\|^2 + \|\Delta\xi\|^2\right)$$

then, there exists C and $C^{-1} > 0$ such that

$$C\| \left(\omega(t), \frac{\partial \omega}{\partial t}(t), \xi(t), \frac{\partial \xi}{\partial t}(t)\right) \|_{\Phi_2}^2 \leqslant E_{14}(t) \tag{6.22}$$

The property of $f(s)$ allows to find, owing to $(u, \alpha) \in B_{R_0} \cap \Phi_2$ and the fact that $u \in H^2(\Omega) \subset L^\infty(\Omega)$, the following estimate

$$\begin{aligned} \|f'(u)\nabla u\|^2 &= \int_\Omega \left(3|u|^2+1\right)^2 |\nabla u|^2 dx \\ &\leqslant C'\left(|u|_{L^\infty}^4 + 1\right)\|\nabla u\|^2 \\ &\leqslant C'\|\nabla u\|^2 \end{aligned}$$

and

$$\begin{aligned} \|f(u)\|^2 &= \int_\Omega \left(u^3 - u\right)^2 |dx \\ &\leqslant \int_\Omega |u|^2 \left(|u|^2+1\right)^2 dx \\ &\leqslant C\left(|u|_{L^\infty}^4 + 1\right)\int_\Omega |u|^2 dx \\ &\leqslant C\|u\|^2 \\ &\leqslant C\|\nabla u\|^2. \end{aligned}$$

Thanks to the estimates (5.2), the two above estimates can be written as following

$$\|f'(u)\nabla u\|^2 \leqslant Q\left(\|(u(0), \frac{\partial \alpha}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\Phi_1}\right) e^{-\beta t} + C \tag{6.23}$$

$$\|f(u)\|^2 \leqslant Q\left(\|(u(0), \frac{\partial \alpha}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\Phi_1}\right) e^{-\beta t} + C \tag{6.24}$$

where β and C are the positive constants and Q is the monotonic function. Inserting (6.23) and (6.24) into (6.20), we have

$$\frac{d}{dt}E_{14} \leqslant Q\left(\|(u(0), \frac{\partial u}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\Phi_1}\right) e^{-\beta t} + C. \tag{6.25}$$

Integrating (6.25) from 0 to $t \in [0, T]$ and owing to (6.22), we get

$$\begin{aligned} \|(\omega(t), \frac{\partial \omega}{\partial t}(t), \xi(t), \frac{\partial \xi}{\partial t}(t))\|_{\phi_2}^2 &\leqslant C\int_0^t \left(Q\left(\|(u(0), \frac{\partial u}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) e^{-\beta\tau} + C\right) d\tau \\ &\leqslant \left(1 + e^{-\beta t}\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\Phi_1}\right) + Ct \\ &\leqslant \left(1 + \frac{1}{e^{\beta t}}\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) + Ct \\ &\leqslant \left(1 + \frac{1}{1+\beta t}\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) + Ct \\ &\leqslant \left(1 + \frac{1}{1+\beta t} + t\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) \\ &\leqslant \left(1 + \beta t + 1 + t + \beta t^2\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) \\ &\leqslant \left(2 + (1+\beta)t + \beta t^2\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}, \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\phi_1}\right) \\ &\leqslant \left(1 + T^2\right) Q\left(\|(u(0), \frac{\partial u}{\partial t}(0), \alpha(0), \frac{\partial \alpha}{\partial t}(0))\|_{\Phi_1}\right). \end{aligned} \tag{6.26}$$

The estimate (6.26) allows to assert that the operator $S_\varepsilon(t)$ is regularizing in Φ_1. Then there exists the global attractor.

7 Conclusion

The works contained in this article relative dynamic system, are very important to explain the processes of phase transition phenomena. Since the solution of the system exists and is unique and the system is dissipative, then the existence of the global attractor associated to the problem (1.5)-(1.8) is proven.

Acknowledgement

The authors wish to thank the Research Group in Functional Analysis and Partial Differential Equations of the faculty of Science and Technology of Mqrien Ngouabi University.

The authors are very grateful to referees for the care that they took in reading this article.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Caginalp G. Conserved phase-fiel system: Implications for kinetic under-cooling. Phys. Rev. B. 1988;38:789-791.

[2] Brochet D, Hilhorst D, Novick-Cohen A. Maximal attractor and inertial sets for a conserved phase-field model. Adv. Differential Equations. 1996;1:547-578.

[3] Brochet D. Maximal attractor and inertial sets for some second and fourth order phase-field models. Pitman Res. Notes Math. Ser. Longman Sci. Tech. Harlow. 1993;296:77-85.

[4] Gilardi G. On a conserved phase-field model with irregular potentials and dynamic boundary condition. Istit. Lombardo. Accad. Sci. Lett. Rend. A. 2007;141:129-161.

[5] Colli P, Gilardi G, Laurenot Ph, Novick-Cohen A. Uniqueness and long-time behavior for the conserved phase-field system melory. Discrete Continuous Dynamical Systems-Seri. 1999;5:375-390.
Available: http://dx.doi.org/10.3934/dcds.1999, 5.375

[6] Colli P, Gilardi G, Grasselli M, Schimperna G. The conserved phase-field system with memory. Adv. Math. Sci Appl. 2001;11:265-291.

[7] Caginalp G. The dynamic of conserved phase-field system: Stefan-Like, Hell-Shaw and Cahn-Hilliard models as asymptotic limits. IMAJ. App. Math. 1990;44:77-94.

[8] Efendiev M, Miranville A, Zelik S. Exponential attractors for a singularly perturbed Cahn-Hilliard system. Math. Nachr. 2004;272:11-31.

[9] S. Gatti, V. Pata. Exponential attractor for a conserved phase-field system with memory. Physica D. 2004;189:31-48.

[10] Miranville Alain. On the conserved phase-field model. J. Math. Anal. Appl. 2013;400:143-152.

[11] Ntsokongo AJ, Batangouna N. Existence and uniqueness of solutions for a conserved phase-field type model. AIMS Mathematics. 2016;1:144-155.
Available: http://dx.doi.org/10.3934/Math.2016.2.144

[12] Miranville Alainm, Quintanilla R. A type III phase-field system with a logarithmic potential. Appl. Maths. Letters. 2011;24:1003-1008.

[13] Miranville Alain, Quintanilla R. A Caginalp phase-field system based on type III heat conduction with two temperatures. Quart Appl. Maths. 2016;74:375-398.

[14] Mangoubi Jean De Dieu, Moukoko D, Moukamba F, Langa FDR.Existence and uniqueness of solutions for Cahn-Hilliard hyperbolic phase-field system with dirichlet boundary condition and regular potentials. Applied Mathematics. 2016;7:1999-1926.
Available:http://dx.doi.org/10.4236/am.2016.716157

[15] Mayeul Evrard I, Moukoko D, Moukamba F, Langa FDR. Existence and uniqueness of solutions for Hyperbolics Field-phase System of Caginalp type with Polynomial growth potential. Internal Mathematical Forum 10. 477-486.

[16] Moukoko Daniel. Well-posedness and longtime behaviors of a hyprebolic caginalp system. Journal of Applied Analysis and Computation. 2014;4(2):151-196.

[17] Moukoko Daniel. Etude des modles hyperboliques de champ de phase de caginalp, these unique, Falcult des Sciences et Techniques. Universit Marien NGOUABI; 2015.

[18] Mayeul Evrard I, Moukoko D, Moukamba F, Langa FDR.Existence of global attractor for hyperbolics field-phase system of caginalp type with polynomial growth potential. Britsh Journal of Mathematics and Computer Science. 2016;18(6):1-18, Article no.BJMCS.28607.

[19] Moukoko Daniel, Moukamba F, Langa FDR. Global attractor for caginalp hyperbolics field-phase system with singular pontential. Journal of Mathematics Research. 2015;7(3):165-177.

[20] Grasselli M, Miranville A, Pata V, Zelik S. Well-posedness and long time behavior of a parabolic-hyperbolics field-phase system with singular pontential. Math. Nachr. 2007;280:1475-1509.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/23715