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Investigation of Zeeman Models with Time Delays

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Authors' contributions

This work was carried out in collaboration between all authors. Author MM designed the study, wrote the protocol and formed the first draft of the paper. Author PO performed the statistical analysis. Authors MM and PO managed the analyses of the study. Author ZB managed the literature searches and the final styling of the manuscript. All authors read and approved the final manuscript.

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Abstract

In this paper, the stability of the equilibrium points of the two dimensional Zeeman Heartbeat Model is investigated with time delay in cardiac muscle fiber or stimulator. The formulation of the heartbeat model is explained in the first part. Then, the stability conditions for the equilibrium points of the system are derived. Finally, some examples are given to illustrate the results of the study. The overall objective of this study is to investigate the effects of time delays on the dynamics of the Zeeman Heartbeat Model.

Keywords: Zeeman Heartbeat model; time delay; stability; equilibrium.

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1 Introduction

The use of mathematical models is becoming increasingly popular in various fields of science. Social sciences, engineering and medicine are just a couple of these areas which mathematical modeling studies

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form a non-negligible part of the studies. In the recent years, models consisting of systems of differential equations have been use to investigate many phenomena such as bacterial resistance [1], dengue disease [2] and tobacco control [3]. The heartbeat models of the British mathematician Sir Erik Christopher Zeeman are further examples of modeling studies in mathematical biology [4,5].

The heart is about the size of a fist. It acts as a pump in the body to control the circulation of blood. It consists of two halves which have different roles for pumping the blood and the heart beats about 3 billion times through the life of a human to regulate the circulation. The oxygen-poor blood, which is dark-red in color, is collected in the right side of the heart, where it is then sent to the lungs. The blood is oxygenated in the lungs. The oxygen-rich blood, which is collected in the left side of the heart, is then sent to the body. The oxygen is used by the organs and then sent back to the heart to complete the cycle. The mechanism of the heart consists of contractions and relaxing. These activities of the heart muscles are controlled by electrical impulses. Systems of ordinary differential equations have been presented by E.C. Zeeman in his studies in 1972 and 1977 to explain the activity of the heart under the control of this electrochemical event [4,5,6].

E. C. Zeeman has used two different deterministic equation systems to model the behavior of the heart. Similar deterministic models are widely used in biochemistry, biology and other fields to analyze various dynamics of the event under investigation. However, deterministic differential equation systems are unable to accurately model some real life events. Hence, the use of random effects, stochastic noise terms and time delays are some of the methods frequently used to improve the accuracy of these models. By introducing time delays in the equation systems, the past states of the system are also considered in the analysis and a more complex investigation of the system can be done [7]. Time delays have been used in predator-prey models [8], epidemic models [9] and also to model population dynamics [10,11,12,13,14,15,16,17,18,19,20, 21]. Additional recent studies on time delays and heartbeat models can also be found in the literature [22,23, 24,25,26,27].

In this study, delay differential equation systems will be obtained by using E.C. Zeeman's model. Time delays will be used in Zeeman's system to model the dynamic behavior of heartbeat from a different perspective. The equation system under time delays will be analyzed for its equilibrium points and the stability of the equilibria. The study proposes a perspective to heartbeat models and the examples in our study can be used to analyze similar equation systems under time delays. The paper is organized as follows: Zeeman's two dimensional heartbeat model is introduced in section 2 and delayed models are given in section 3 along with some results.

2 Zeeman's Heartbeat Model

E.C. Zeeman's heartbeat model consisting of two differential equations is given as follows [4]

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{1}{\varepsilon}\left(x_1(t)^3 - Tx_1(\mathrm{t}) + x_2(t)\right), \\ \frac{dx_2}{dt} &= x_1(t) - x_d. \end{aligned} \tag{1}$$

The variables of the system are x_1 and x_2 which denote the muscle fiber length and the stimulus that controls muscle fiber contraction at any time t, respectively. Parameters of system (1) have been acquired from the referred study [4] and are listed in Table 1.

Table 1. Values and descriptions of the parameters

Parameter	Description	Value
ε	A constant dependent on the timescale	0.2
T	Overall tension in the heart	0.5
x_d	Typical relaxed fiber length	0 or 0.41

Initial values of the variables are given as [4]:

$$x_1(0) = 0.5; x_2(0) = 0.$$

3 Two Dimensional Zeeman Model with Time Delays

3.1 First case

If the time delay, denoted by τ, is introduced in the second term of the first equation in (1), we obtain:

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{1}{\varepsilon}(x_1(t)^3 - Tx_1(\mathrm{t}) + x_2(t-\tau)), \\ \frac{dx_2}{dt} &= x_1(t) - x_d. \end{aligned} \tag{2}$$

Let $x_d \neq 0$. Assume $Q(x_{1_0}, x_{2_0})$ be a positive equilibrium point of (2). We linearize the system (2) at the equilibrium point Q as follows:

Assume $u(t) = x_1(t) - x_{1_0}$, $v(t) = x_2(t) - x_{2_0}$. By using the change of variables

$$\frac{du}{dt} = \frac{dx_1}{dt}, \qquad \frac{dv}{dt} = \frac{dx_2}{dt}$$

and by choosing the functions f and g as

$$f(x_1, x_2) = -\frac{1}{\varepsilon}(x_1(t)^3 - Tx_1(\mathrm{t}) + x_2(t-\tau))$$

$$g(x_1, x_2) = (x_1(\mathrm{t}) - x_d),$$

we can linearize the system (2) at the point $Q(x_{1_0}, x_{2_0})$ as

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial f}{dx_1}(Q^*)u(t) + \frac{\partial f}{dx_2}(Q^*)v(t-\tau) \\ \frac{dv}{dt} &= \frac{\partial g}{dx_1}(Q^*)u(t) + \frac{\partial g}{dx_2}(Q^*)v(t), \end{aligned} \tag{3}$$

which becomes

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{\varepsilon}\left(3x_{1_0}^2 - T\right)u(t) + \left(-\frac{1}{\varepsilon}\right)v(t-\tau) \\ \frac{dv}{dt} &= u(t). \end{aligned} \tag{4}$$

The characteristic equation of the system (4) is

$$\Delta(\lambda,\tau) = \det\left(\lambda I - A_0 - \Sigma_{j=1}^{1} A_j e^{-\lambda\tau j}\right),$$

where

$A_0 = \begin{pmatrix} -\frac{3}{\varepsilon}x_{1_0}^2 + \frac{T}{\varepsilon} & 0 \\ 1 & 0 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -1/\varepsilon \\ 0 & 0 \end{pmatrix}$. Hence,

$$\Delta(\lambda,\tau) = \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -\frac{3}{\varepsilon}x_{1_0}^2 + \frac{T}{\varepsilon} & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1/\varepsilon \\ 0 & 0 \end{pmatrix} e^{-\lambda\tau}\right) = \det\begin{pmatrix} \lambda + \frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon} & \frac{1}{\varepsilon}e^{-\lambda\tau} \\ -1 & \lambda \end{pmatrix}.$$

$$\mathrm{P}(\lambda) = \lambda^2 + \left(\frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon}\right)\lambda + \frac{1}{\varepsilon}e^{-\lambda\tau} = 0 \tag{5}$$

$$\Rightarrow \mathrm{P}(\lambda) = \lambda^2 + a\lambda + ce^{-\lambda\tau}. \tag{6}$$

Let,

$$a = \frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon}, c = \frac{1}{\varepsilon}.$$

The characteristic equation (6) for $\tau = 0$ is

$$\lambda^2 + a\lambda + c = 0 \tag{7}$$

and the roots of (8) are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4c}}{2}. \tag{8}$$

(H1) For $a = 0$, $\lambda_1 = -i\sqrt{c}$ and $\lambda_2 = i\sqrt{c}$. Then,

$$\lambda_{1,2} = \pm i\sqrt{c}$$

Thus, $Q(x_{1_0}x_{2_0})$ becomes a steady center point for $a = 0$.

(H2) For $a < 0$, the eigenvalues $\lambda_1 = \frac{-a-\sqrt{a^2-4c}}{2}$ and $\lambda_2 = \frac{-a-\sqrt{a^2-4c}}{2}$ are found. Hence for $a = 0$, once again $\lambda_{1,2} = \pm i\beta, \beta \neq 0$ is obtained.

Thus, if we generalize these cases for $a \in R$; the equilibrium point Q becomes a steady center point for $a \in R$, since $\lambda_{1,2} = \pm i\beta, \beta \neq 0$.

All roots of Eq. (7) have negative real parts, which is true if

(H3) $a > 0$

(H4) $c > 0$

Assume that $\tau \neq 0$ in the characteristic equation (6). Let $\lambda = i\overline{w}, \overline{w} > 0$ be a root of the characteristic equation (6). Writing the values of λ in the equation (6),

$$e^{-\lambda\tau} = e^{-i\overline{w}\tau} = \cos(\overline{w}\tau) - isin(\overline{w}\tau)$$

can be obtained. Thus,

$$(i\overline{w})^2 + ai\overline{w} + c[\cos(\overline{w}\tau) - isin(\overline{w}\tau)] = 0.$$

From here, we find that

$$-\overline{w}^2 + ai\overline{w} + \mathrm{c}\cos(\overline{w}\tau) - cisin(\overline{w}\tau) = 0 \quad (9)$$

The real and imaginary parts of the equation (9) can be handled separately as

$$-\overline{w}^2 + \mathrm{c}\cos(\overline{w}\tau) = 0 \quad (10)$$

and

$$a\overline{w} - \mathrm{c}sin(\overline{w}\tau) = 0. \quad (11)$$

After appropriate arrangements, we find that

$$\cos(\overline{w}\tau) = \frac{\overline{w}^2}{c}, sin(\overline{w}\tau) = \frac{a\overline{w}}{c}. \quad (12)$$

Square both sides of equations (12) respectively, then we get

$$\overline{w}^4 + a^2\overline{w}^2 - c^2 = 0 \quad (13)$$

The roots of equation (13) are

$$\overline{w}^2 \pm = \frac{-a^2 \pm \sqrt{a^4 + 4c^2}}{2} \quad (14)$$

Thus equation (14) has one positive and one negative roots. Now, we are going to determine the values τ_k as follows:

Dividing the equations of (12) yields

$$\tan(\overline{w}\tau_k) = \frac{sin(\overline{w}\tau)}{\cos(\overline{w}\tau)} = \frac{\frac{a\overline{w}}{c}}{\frac{\overline{w}^2}{c}} = \frac{a}{\overline{w}} \quad (15)$$

$\tan^{-1}$ of both sides in the equation (15) yields

$$\overline{w}\tau_k = \tan^{-1}\left(\frac{\mathrm{a}}{\overline{w}}\right) + 2\mathrm{k}\pi, \mathrm{k} = 0,1,2 \ldots \quad (16)$$

$$\tau_k = \frac{1}{\overline{w}}\tan^{-1}\left(\frac{\mathrm{a}}{\overline{w}}\right) + \frac{2\mathrm{k}\pi}{\overline{w}} \quad k = 0,1,2 \ldots. \quad (17)$$

Example: Considering the system (2) for the parameters $\varepsilon = 0.2, T = 0.5, x_d = 0.41$, we find that the positive equilibrium point of the system (2) is $(0.41, 0.136079)$. For $\tau = 0$, the eigenvalues of the jacobian matrix of the system (2) at the point $(0.41, 0.136079)$ are $\{-0.0107 + 2.2360i,\ -0.0107 - 2.2360i\}$. Noticing that the real parts of the complex conjugate eigenvalues are negative, $(0.41, 0.136079)$ becomes a stable equilibrium point.

For $\tau = 0$, we find that $a = 0.0215$. The values of $\overline{w}_{+}^{2}$ and τ can not be determined by using a.

3.2 Second case

If the time delay is introduced in the first term of the second equation in (1), we obtain:

$$\frac{dx_1}{dt} = -\frac{1}{\varepsilon}(x_1(\mathrm{t})^3 - Tx_1(\mathrm{t}) + x_2(t))$$
$$\frac{dx_2}{dt} = x_1(t-\tau) - x_d. \tag{18}$$

Here, let $x_d \neq 0$.

Assume $Q(x_{1_0}, x_{2_0})$ be a positive equilibrium point of (18). We linearize the system (18) at the equilibrium point Q as follows [28]: Assume $u(t) = x_1(t) - x_{1_0}, v(t) = x_2(t) - x_{2_0}$, then $\frac{du}{dt} = \frac{dx_1}{dt}$ and $\frac{dv}{dt} = \frac{dx_2}{dt}$. Assume we choose

$$f(x_1, x_2) = -\frac{1}{\varepsilon}(x_1(t)^3 - Tx_1(\mathrm{t}) + x_2(t))$$

and

$$g(x_1, x_2) = x_1(t-\tau) - x_d.$$

Thus, by linearizing the system (18) at the point $Q(x_{1_0}, x_{2_0})$:

$$\frac{du}{dt} = \frac{\partial f}{dx_1}(Q^*)u(t) + \frac{\partial f}{dx_2}(Q^*)v(t)$$
$$\frac{dv}{dt} = \frac{\partial g}{dx_1}(Q^*)u(t-\tau) + \frac{\partial g}{dx_2}(Q^*)v(t), \tag{19}$$

which after necessary arrangements becomes the system

$$\frac{du}{dt} = -\frac{1}{\varepsilon}\left(3x_{1_0}^2 - T\right)u(t) + \left(-\frac{1}{\varepsilon}\right)v(t)$$
$$\frac{dv}{dt} = u(t-\tau). \tag{20}$$

The characteristic equation of the system (20) is

$$\Delta(\lambda, \tau) = \det\left(\lambda I - A_0 - \Sigma_{j=1}^{1} A_j e^{-\lambda\tau j}\right),$$

where

$$A_0 = \begin{pmatrix} -\frac{3}{\varepsilon}x_{1_0}^2 + \frac{T}{\varepsilon} & -1/\varepsilon \\ 0 & 0 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \text{ Hence,}$$

$$\Delta(\lambda,\tau) = \det\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} - \begin{pmatrix}-\frac{3}{\varepsilon}x_{1_0}^2 + \frac{T}{\varepsilon} & -1/\varepsilon\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\mathrm{e}^{-\lambda\tau}\right) = \det\begin{pmatrix}\lambda + \frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon} & \frac{1}{\varepsilon}\\ -\mathrm{e}^{-\lambda\tau} & \lambda\end{pmatrix}.$$

$$\mathrm{P}(\lambda) = \lambda^2 + \left(\frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon}\right)\lambda + \frac{1}{\varepsilon}\mathrm{e}^{-\lambda\tau} = 0 \tag{21}$$

$$\Rightarrow \mathrm{P}(\lambda) = \lambda^2 + a\lambda + c\mathrm{e}^{-\lambda\tau} = 0. \tag{22}$$

where we choose

$$a = \frac{3}{\varepsilon}x_{1_0}^2 - \frac{T}{\varepsilon}, \qquad c = \frac{1}{\varepsilon}.$$

The equation (6) was obtained in a manner similar to the above operations.

The characteristic equation (22) for $\tau = 0$ is

$$\lambda^2 + a\lambda + \mathrm{c} = 0 \tag{23}$$

and the roots of (23) are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4\mathrm{c}}}{2}. \tag{24}$$

All roots of Eq. (23) have negative real parts, if

(H5) $a > 0$

(H6) $c > 0$

the equilibrium point Q becomes locally asymptotically stable when both conditions hold.

We want to determine if the real part of some root increases to reach zero and eventually becomes positive as τ varies.

Assume $\tau \neq 0$ in the characteristic equation (22). Let $\lambda = i\overline{w}, \overline{w} > 0$ be a root of the characteristic equation (22). For

$$e^{-\lambda\tau} = e^{-i\overline{w}\tau} = \cos(\overline{w}\tau) - i\sin(\overline{w}\tau),$$

If λ is written in (22), we obtain

$$(i\overline{w})^2 + ai\overline{w} + [\cos(\overline{w}\tau) - i\sin(\overline{w}\tau)]c = 0.$$

Hence

$$-\overline{w}^2 + ai\overline{w} + \cos(\overline{w}\tau)\,\mathrm{c} - i\sin(\overline{w}\tau)c = 0 \tag{25}$$

The real and imaginary parts of the equation (25) can be handled separately as

$$-\overline{w}^2 + \cos(\overline{w}\tau)\,c = 0 \Rightarrow \cos(\overline{w}\tau) = \frac{\overline{w}^2}{c} \tag{26}$$

and

$$a\overline{w} - sin(\overline{w}\tau)\mathrm{c} = 0 \Rightarrow sin(\overline{w}\tau) = \frac{a\overline{w}}{c}. \tag{27}$$

Thus,

$$\cos(\overline{w}\tau) = \frac{\overline{w}^2}{c} \tag{28}$$

$$sin(\overline{w}\tau) = \frac{a\overline{w}}{c}. \tag{29}$$

By taking the square of both sides in (28) and (29) and adding these equations side to side;

$$\begin{cases} \dfrac{\overline{w}^{2^2}}{c^2} = \cos^2 \overline{w}\tau \\ \dfrac{a^2\overline{w}^2}{c^2} = \sin^2 \overline{w}\tau \\ (\overline{w}^{2^2} + a^2\overline{w}^2)\dfrac{1}{c^2} = 1 \end{cases}$$

$$t^2 + a^2 t - c^2 = 0 \tag{30}$$

is obtained. From here we get

$$t_{1,2} = \frac{-a^2 \pm \sqrt{\mathrm{a}^4 + 4c^2}}{2}$$

$$\overline{w}_{\pm}^2 = \frac{-a^2 \pm \sqrt{\mathrm{a}^4 + 4c^2}}{2}. \tag{31}$$

(H7) For any a value, $\overline{w}_{\pm}^2$ has two complex conjugate roots, two of which are real.

Let's determine the values of τ_k as below. If the equations (28)-(29)

$$\cos(\overline{w}\tau) = \frac{\overline{w}^2}{c}$$

$$sin(\overline{w}\tau) = \frac{a\overline{w}}{c}$$

are divided side by side;

$$\tan(\overline{w}\tau_k) = \frac{a}{\overline{w}}. \tag{32}$$

The compound of $\tan^{-1}$ with both sides in the equation (32) yields

$$\overline{w}\tau_k = \tan^{-1}\left(\frac{a}{\overline{w}}\right) + 2k\pi, \qquad k = 0,1,2,3 \dots. \tag{33}$$

The equation (33) is:

$$\tau_k = \frac{1}{\overline{w}} \tan^{-1}\left(\frac{a}{\overline{w}}\right) + \frac{2k\pi}{\overline{w}^2}, \quad k = 0,1,2,3 \ldots \tag{34}$$

$$\tau_k^{\pm} = \frac{1}{\overline{w}_{\pm}} \tan^{-1}\left(\frac{a}{\overline{w}}\right) + \frac{2k\pi}{\overline{w}_{\pm}{}^2}, \quad k = 0,1,2,3 \ldots. \tag{35}$$

The above discussion can be summarized by the following lemma. The proof of this lemma can also be found in [29].

Lemma 1:

i) If **(H5)** and **(H6)** are satisfied, then the equation (22), with $\tau = \tau_j^+$, only has imaginary roots $\pm \mathrm{i}\overline{\mathrm{w}}$.
ii) If **(H5)** and **(H6)** hold and $\tau = \tau_j^-$, then the equation (22) only has the imaginary root couple $\pm \mathrm{i}\overline{\mathrm{w}}$.

Let's introduce the following theorem to determine the stability of the equation with time delay.

Theorem 1: If the following conditions are satisfied, the equilibrium point Q^* is asymptotically for all $\tau \geq 0$.

i) All of the real parts of the roots of the equation $\Delta(\lambda, 0) = 0$ are negative.
ii) For all real $(\overline{w})$ and $\tau \geq 0$; $\Delta(i\overline{w}, \tau) \neq 0, \mathrm{i} = \sqrt{-1}$.

Now as a result, we obtain the following theorem.

Theorem 2: If the conditions hold for $a < 0\,, c > 0$, the equilibrium point Q of the equation (22) is asymptotically stable for all $\tau \geq 0$.

For the proof of the contradictory conditions;

$$\frac{d}{d\tau} Re(\lambda_j^+(\tau_j^+)) > 0 \text{ and } \frac{d}{d\tau} Re(\lambda_j^-(\tau_j^-)) < 0$$

The derivative of the both sides of the equation (22) which depends on τ:

$$\lambda^2 + a\lambda + c\mathrm{e}^{-\lambda\tau} = 0$$

$$\Rightarrow 2\lambda \frac{d\lambda}{d\tau} + a\frac{d\lambda}{d\tau} - \mathrm{ce}^{-\lambda\tau}\left[\tau\frac{d\lambda}{d\tau} + \lambda\right] = 0$$

$$\Rightarrow \left[2\lambda + \mathrm{a} - \mathrm{ce}^{-\lambda\tau}\tau\right]\frac{d\lambda}{d\tau} - \mathrm{c}\lambda\mathrm{e}^{-\lambda\tau} = 0$$

$$\frac{d\lambda}{d\tau} = \frac{c\lambda\mathrm{e}^{-\lambda\tau}}{2\lambda + a - \mathrm{ce}^{-\lambda\tau}\tau}. \tag{36}$$

Since $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{\frac{d\lambda}{d\tau}}$,

$$\frac{d\tau}{d\lambda} = \frac{2\lambda + a - \mathrm{ce}^{-\lambda\tau}\tau}{c\lambda\mathrm{e}^{-\lambda\tau}}. \tag{37}$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a - \mathrm{ce}^{-\lambda\tau}\tau}{c\lambda\mathrm{e}^{-\lambda\tau}} = \frac{2\lambda + a}{c\lambda\mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{38}$$

$\lambda^2 + a\lambda + c\mathrm{e}^{-\lambda\tau} = 0$ from equation, $\mathrm{e}^{-\lambda\tau} = \frac{-\lambda^2 - a\lambda}{c}$ (39)

Substitution of equation (39) into equation (38), yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + a - \mathrm{ce}^{-\lambda\tau}\tau}{c\lambda\mathrm{e}^{-\lambda\tau}} = \frac{-\mathrm{a} - 2\lambda}{\lambda(\lambda^2 + a\lambda)} - \frac{\tau}{\lambda}$$

$$\mathrm{sign}\left\{\frac{d}{d\tau}(Re\lambda)\right\} = \mathrm{sign}\left\{Re(\left(\frac{d\lambda}{d\tau}\right)^{-1})\right\}.$$

$$\mathrm{sign}\left\{\frac{d\big((Re\lambda)\big)}{d\tau}\bigg|_{\lambda=i\overline{w}}\right\} = \mathrm{sign}\left\{Re\left(\left(\frac{d\lambda}{d\tau}\right)^{-1}\right)\bigg|_{\lambda=i\overline{w}}\right\}.$$

Hence, the expression

$$\mathrm{sign}\left\{\frac{d\big((Re\lambda)\big)}{d\tau}\bigg|_{\lambda=i\overline{w}}\right\} = \mathrm{sign}\left\{\left(\frac{-\mathrm{a} - 2\lambda}{\lambda(\lambda^2 + a\lambda)} - \frac{\tau}{\lambda}\right)\bigg|_{\lambda=i\overline{w}}\right\} \tag{40}$$

can be calculated as below. Using $i^2 = -1$, the calculation yields:

$$\mathrm{sign}\left\{\frac{d\big((Re\lambda)\big)}{d\tau}\bigg|_{\lambda=i\overline{w}}\right\} = sign\left\{Re\left(\left(\frac{-\mathrm{a} - 2\lambda}{\lambda(\lambda^2 + a\lambda)}\right)\bigg|_{\lambda=i\overline{w}}\right) - Re\left(\frac{\tau}{\lambda}\right)\bigg|_{\lambda=i\overline{w}}\right\}$$

Now, we evaluate the following to compute $\mathrm{sign}\left\{\frac{d\left((Re\lambda)\right)}{d\tau}\Big|_{\lambda=i\overline{w}}\right\}$:

1. The value of $Re\left(\left(\frac{-\mathrm{a}-2\lambda}{\lambda(\lambda^2+a\lambda)}\right)\Big|_{\lambda=i\overline{w}}\right)$ is as follows:

$$Re\left(\frac{-\mathrm{a} - 2i\overline{w}}{i\overline{w}(-\overline{w}^2 + ai\overline{w})}\right) = Re\left(\frac{-\mathrm{a}}{i\overline{w}(-\overline{w}^2 + ai\overline{w})}\right) + Re\left(\frac{-2i\overline{w}}{i\overline{w}(-\overline{w}^2 + ai\overline{w})}\right)$$

But,

$$Re\left(\frac{-2i\overline{w}}{i\overline{w}(-\overline{w}^2 + ai\overline{w})}\right) = Re\left(\frac{-2i\overline{w}}{-i\overline{w}(\overline{w}^2 + ai\overline{w})}\right) = Re\left(\frac{2}{(\overline{w}^2 - ai\overline{w})}\frac{(\overline{w}^2 + ai\overline{w})}{(\overline{w}^2 + ai\overline{w})}\right) = Re\left(\frac{2\overline{w}^2 + 2ai\overline{w}}{(\overline{w}^4 + a^2\overline{w}^2)}\right)$$
$$= \frac{2\overline{w}^2}{(\overline{w}^4 + a^2\overline{w}^2)}$$

$$Re\left(\frac{-\mathrm{a}}{i\overline{w}(-\overline{w}^2+ai\overline{w})}\right) = Re\left(\frac{\mathrm{a}}{i\overline{w}(\overline{w}^2-ai\overline{w})}\right) = Re\left(\frac{\mathrm{a}}{i\overline{w}(\overline{w}^2-ai\overline{w})}\frac{(\overline{w}^2+ai\overline{w})}{(\overline{w}^2+ai\overline{w})}\right) = Re\left(\frac{\mathrm{a}\overline{w}^2+a^2i\overline{w}}{i\overline{w}(\overline{w}^4+a^2\overline{w}^2)}\right) = \frac{\mathrm{a}^2}{(\overline{w}^4+a^2\overline{w}^2)}$$

Therefore,

$$Re\left(\left(\frac{-\mathrm{a} - 2\lambda}{\lambda(\lambda^2 + a\lambda)}\right)\bigg|_{\lambda=i\overline{w}}\right) = \frac{2\overline{w}^2 + a^2}{(\overline{w}^4 + a^2\overline{w}^2)} \tag{41}$$

2. The value of $Re\left(\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\overline{w}}$ is as follows:

$$Re\left(\frac{\tau}{\lambda}\right)\Big|_{\lambda=i\overline{w}} = Re\left(\frac{\tau}{i\overline{w}}\frac{-i\overline{w}}{-i\overline{w}}\right) = Re\left(\frac{-\tau i\overline{w}}{\overline{w}^2}\right) = 0 \tag{42}$$

By substituting equations (41) and (42) into equation (40), we arrive at

$$\text{sign}\left\{\frac{d\big((Re\lambda)\big)}{d\tau}\Big|_{\lambda=i\overline{w}}\right\} = sign\left\{\frac{2\overline{w}^2 + a^2}{(\overline{w}^4 + a^2\overline{w}^2)}\right\} \tag{43}$$

Theorem 3: Let $\tau = \tau^{\pm}$ be defined as in (35). If the conditions hold;

$$\lambda^2 + a\lambda + c\mathrm{e}^{-\lambda\tau} = 0,$$

then the equilibrium point Q is unstable for a positive constant m

$$\tau\epsilon[0,\tau_0^+] \cup [\tau_0^-,\tau_1^+] \cup \ldots \cup [\tau_{m-1}^-,\tau_m]$$

$$\tau\epsilon[\tau_0^+,\tau_0^-] \cup [\tau_1^+,\tau_1^-,] \cup \ldots \cup [\tau_{m-1}^+,\tau_{m-1}^-].$$

Proof: When the following conditions hold for the theorem, then only the following contradictory conditions need to be satisfied. By

$$\frac{d}{d\tau}(Re\ \lambda)|_{\tau=\tau^+} > 0 \text{ and } \frac{d(Re\ \lambda)}{d\tau}|_{\lambda=i\overline{w}^+} > 0,$$

$$\frac{d}{d\tau}(Re\ \lambda)|_{\tau=\tau^-} > 0 \text{ and } \frac{d(Re\ \lambda)}{d\tau}|_{\lambda=i\overline{w}^-} < 0,$$

$$\overline{w}_\pm^2 = \frac{-a^2 \pm \sqrt{\mathrm{a}^4 + 4c^2}}{2}$$

and

$$\tau_k^\pm = \frac{1}{\overline{w}_\pm}\tan^{-1}\left(\frac{a}{\overline{w}}\right) + \frac{2k\pi}{\overline{w}_\pm{}^2}, \quad k = 0,1,2,3 \ldots.$$

from the conditions of Theorem 3,

$$\mathrm{a}^4 + 4c^2 > 0$$

$$\sqrt{\mathrm{a}^4 + 4c^2} > 0$$

$$\text{sign}\left\{\sqrt{\mathrm{a}^4 + 4c^2)}\right\} > 0.$$

Thus,

$$\text{sign}\left\{\frac{d}{d\tau}(Re\ \lambda)|_{\lambda=i\overline{w}^+}, \tau = \tau^+\right\} > 0.$$

Now,

$$sign\left\{\frac{d}{d\tau}(Re\ \lambda)|_{\lambda=i\overline{w}}\right\} = sign\left\{\frac{2\overline{w}^2 + a^2}{(\overline{w}^4 + a^2\overline{w}^2)}\right\} = sign\left\{\sqrt{\mathrm{a}^4 + 4c^2)}\right\} > 0.$$

Which implies that

$$\text{sign}\left\{-\sqrt{\text{a}^4 + 4c^2)}\right\} < 0.$$

Therefore,

$$\text{sign}\left\{\frac{d}{d\tau}(Re\ \lambda)|_{\lambda=i\overline{w}^+}, \tau = \tau^-\right\} < 0.$$

Hence, the contradictory conditions are satisfied and the proof is completed.

Example: If the system (18) is considered for the parameters $\varepsilon = 0.2, T = 0.5, x_d = 0.41$, the positive equilibrium point of system (18) becomes $(0.41, 0.136079)$. For $\tau = 0$, the eigenvalues of the Jacobian matrix of system (18) at the point $(0.41, 0.136079)$ are obtained as $\{-0.7482 + 2.1072i,\ -0.7482 - 2.1072i\}$. Noticing that the real parts of the complex conjugate eigenvalues are negative, $(0.41, 0.136079)$ becomes a stable equilibrium point.

Since $\tau = 0$, we find that

$$a = 0.0215,$$

$$c = 5.$$

If the values of a and c are put into their places in (31),

$$\overline{w}_+^2 = 4.9988,$$

$$\overline{w}_-^2 = -5.0002.$$

From the square-root of $\overline{w}_+^2$,

$$\overline{w}_+ = \sqrt{4.9988} = 2.2357.$$

If the values of a, c and $\overline{w}_+$ are put into their places in (33), then the value of the first delay term becomes

$$\tau_0^+ = 0.011,$$

$$\tau_1^+ = 1.2581.$$

The critical value of time delay is $\tau = \tau_0^+ = 0.0215$. When $\tau < 0.011$, the equilibrium point $(0.41, 0.136079)$ becomes asymptotically stable; when $\tau = 0.0215$, the stability at $(0.41, 0.136079)$ is lost and when $\tau > 0.011$, the equilibrium point $(0.41, 0.136079)$ becomes unstable.

4 Conclusion

E. C. Zeeman's heartbeat model has been analyzed under time delays in two different terms of the model. The equilibrium points of these models with time delays have been investigated along with their stabilities. Examples have also been given for the time delays, showing the critical values of the time delays for the stability of the equilibrium points. The phase portraits of the model for under different time delays are given in Figs. 1 and 2 to visualize the behavior of the solution curves for the variables $x_1(t)$ and $x_2(t)$. Similar mathematical models can also be investigated with time delays using this approach.

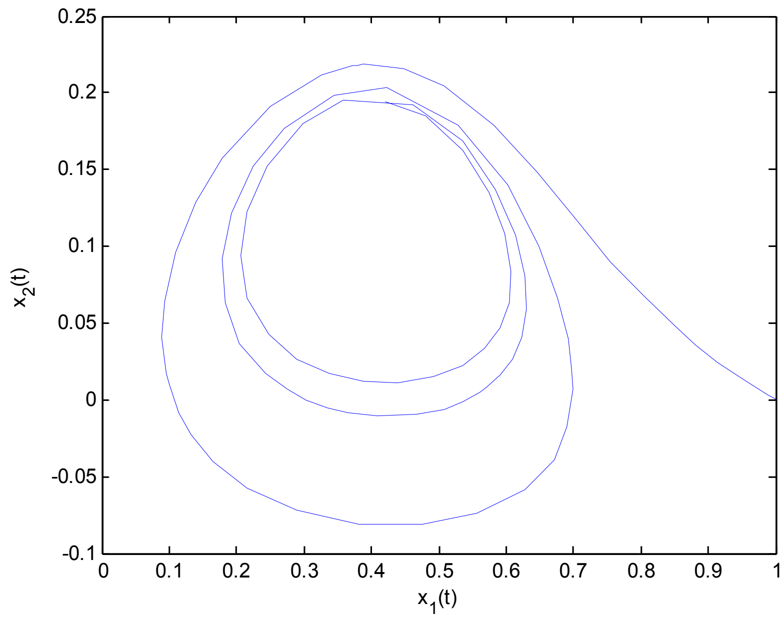

Fig. 1. Solution curve with $\tau_0^+ = 0.011$.

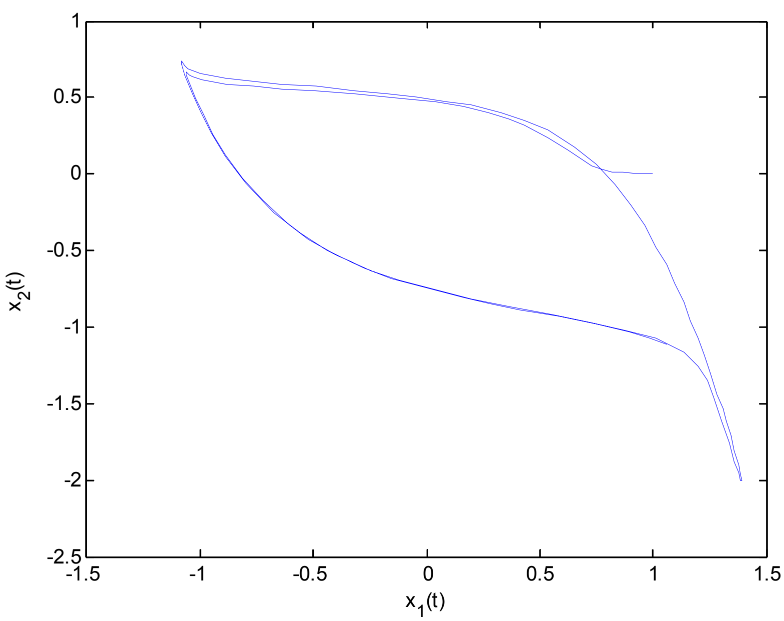

Fig. 2. Solution curve with $\tau_1^+ = 1.2581$

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Merdan M, Bekiryazici Z, Kesemen T, Khaniyev T. Comparison of stochastic and random models for bacterial resistance. Advances in Difference Equations. 2017;133.
DOI: 10.1186/s13662-017-1191-5

[2] Bekiryazici Z, Merdan M, Kesemen T, Najmuldeen M. Mathematical modeling of dengue disease under random effects. Mathematical Sciences and Applications E-Notes. 2016;4(2):58-70.

[3] Pang L, Zhao Z, Liu S, Zhang X. A mathematical model approach for tobacco control in China. Applied Mathematics and Computation. 2015;259:497-509.
DOI: 10.1016/j.amc.2015.02.078

[4] Zeeman EC. Differential equations for the heartbeat and nerve impulse. Coventry, UK: Mathematics Institute, University of Warwick; 1972.

[5] Zeeman EC. Catastrophe theory. Selected Papers 1972–1977. Reading, MA: Addison-Wesley; 1977.

[6] Stewart I, Woodcock AE. On Zeeman's equations for the nerve impulse. Bulletin of mathematical biology. 1981;43(3):279-325.

[7] Lin G, Hong Y. Delay induced oscillation in predator–prey system with Beddington–DeAngelis functional response. Applied Mathematics and Computation. 2007;190(2):1296–1311.

[8] Kar TK, Ghorai A. Dynamic behaviour of a delayed predator–prey model with harvesting. Applied Mathematics and Computation. 2011;217(22):9085-9104.

[9] Zhang FF, Jin Z, Sun GQ. Bifurcation analysis of a delayed epidemic model. Applied Mathematics and Computation. 2010;216(3):753-767.

[10] Mao X, Yuan C, Zou J. Stochastic differential delay equations of population dynamics. J. Math. Anal. Appl. 2005;304(1):296–320.

[11] McSharry PE, Clifford G, Tarassenko L, Smith LA. Method for generating an artificial RR tachogram of a typical healthy human over 24-hours. Computers in Cardiology. 2002;29:225–228.

[12] Malik M, Camm AJ. Heart rate variability. Clinical Cardiology 1990;13(8):570-576.

[13] Bonow RO, Mann DL, Zipes DP, Libby R. Braunwald’s Heart Disease E-Book: a textbook of cardiovascular medicine. Elsevier Health Sciences; 2011.

[14] Suckley R, Biktashev VN. Comparison of asymptotics of heart and nerve excitability. Phys. Rev. E 2003;68(1):011902.

[15] Tu Pierre NV. Dynamical systems—an introduction with applications in economics and biology, 2e. Berlin: Springer; 1994.

[16] Zeeman EC. Differential equations for the heartbeat and nerve impulse. In: Waddington, C.H. (Ed.), Towards a Theoretical Biology 4: Essays, 8–67, Edinburgh University Press; 1972.

[17] Brauer F, Soudack AC. Stability regions and transition phenomena for harvested predator-prey systems. J. Math. Biol. 1979;7(4):319-337.

[18] Myerscough MR, Gray BF, Hogarth WL, Norbury J. An analysis of an ordinary differential equation model for a two-species predator-prey system with harvesting and stocking. J. Math. Biol. 1992;30(4):389-411.

[19] Dai G, Tang M. Coexistence region and global dynamics of a harvested predator-prey system. SIAM J. Appl. Math. Biol. 1998;58(1):193-210.

[20] Xiao D, Ruan S. Bogdanov-Takens bifurcations in predator-prey systems with constant rate of harvesting. Fields Institute Communications. 1999;21:493-506.

[21] Martin A, Ruan S. Predator-prey models with delay and prey harvesting, J. Math. Biol. 2001;43(3):247-267.

[22] Sargolzaei A, Yen K, Abdelghani MN. Delayed inputs attack on load frequency control in smart grid. In Innovative smart grid technologies conference (ISGT), 2014 IEEE PES (pp. 1-5). IEEE; 2014.

[23] Sargolzaei A, Yen KK, Abdelghani MN. Preventing time-delay switch attack on load frequency control in distributed power systems. IEEE Transactions on Smart Grid. 2016;7(2):1176-1185.

[24] Sargolzaei A, Yen K, Abdelghani, MN. Control of nonlinear heartbeat models under time-delay-switched feedback using emotional learning control. International Journal on Recent Trends in Engineering & Technology. 2014;10(2):85.

[25] Thanom W, Loh RNK. Nonlinear control of heartbeat models. Journal on systemics, Cybernetics and Informatics; 2011;9(1):21-27.

[26] Paez-Hernandez R, Zamora-Gómez M, Portillo-Díaz P, Ladino-Luna D, Luevano-Enríquez R. Effects of time delays in a dynamical heart model. Journal of Chemistry and Chemical Engineering. 2014;8(3):320-323.

[27] MacDonald N, MacDonald N. Biological delay systems: linear stability theory (Vol. 9). Cambridge University Press; 2008.

[28] Dieudonné J. Foundations of modern analysis. 2nd Ed. Academic Press; 1969.

[29] Cushing JM, Saleem M. Predator prey model with age structure. Journal of Mathematical Biology. 1982;14(2):231-250.

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